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# Fixed point theorem for Nonlinear $\theta-\phi-$contraction via $w-$distance

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**Abstract**

This paper is aimed to the notion of $\theta-\phi-$contraction defined on a metric space with $w-$distance. Moreover, fixed point theorems are given in this framework. Some illustrative examples are provided to advocate the usability of our results. As an application, we prove the existence and uniqueness of a solution for the nonlinear Fredholm integral equations.

**Keywords:** Fixed point, Nonlinear $\theta-\phi-$contraction, $w-$distance, integral equation.

## 1. Introduction

By a contraction on a metric space $(X, d)$, we understand a mapping $T : X \to X$ satisfying for all $x, y \in X$: $d(Tx, Ty) \leq kd(x, y)$, where $k$ is a real in $[0, 1)$.

In 1922 Banach proved the following theorem.

**Theorem** [2]. Let $(X, d)$ be a complete metric space. Let $T : X \to X$ be a contraction. Then:

(i) $T$ has a unique fixed point $x \in X$.

(ii) For every $x_0 \in X$, the sequence $(x_n)$, where $x_{n+1} = Tx_n$, converges to $x$.

(iii) We have the following estimate: $d(x_n, x) \leq \dfrac{k^n}{1-k} d(x_0, x_1)$, $n \in \mathbb{N}$.

As the result of its intelligibility and profitableness, the previous theorem has become a very celebrated and popular tool in solving the existence problems in many branches of mathematical analysis.

Many mathematicians extended the Banach contraction principle in two major directions, one by stating the conditions on the mapping $T$ and second taking the set $X$ as more general structure [4, 10, 9, 5].

In 2014 Jleli et al. [11] introduced the concept of $\theta-$contraction, using this concept, he proved the existence and uniqueness of a fixed point in complete rectangular metric spaces. This direction has been studied and generalized in different spaces and various fixed point theorems are developed [6, 7, 8].

In [13], D. Zheng has proved fixed point theorem for $\theta-\phi-$contraction, which is perceived to be one of the most general non-linear contraction in complete metric spaces.

In 1996 Kada et al. [3] initiated the notion of $w$-distance on a metric space, then many authors used this concept to prove some results on fixed point theory [1, 5].

Recently Wongyat and Sintunavarat [12] introduced a special $w$-distance called ceiling distance and proved some fixed point theorems for generalized contraction mappings with respected to this distance.

In this paper, we shall obtain a fixed point theorem for $\theta-\phi-$contraction with respect to $w$-distance on complete metric spaces. Various examples are constructed to illustrate our results. As an application, we prove the existence and uniqueness of a solution for the nonlinear Fredholm integral equations.

## 2. Preliminaries

Kada et al. [3] introduced the concept of $w-$distance on a metric space as follows:

**Definition 0.1.** [3] Let $(X, d)$ be a metric space. A function $q : X \times X \to \mathbb{R}^+$ is called a $w$-distance on $X$, if it satisfies the following three conditions for all $x, y, z \in X$ :

- $(W_1)$ $q(x, y) \leq q(x, z) + q(z, y)$;
- $(W_2)$ $q(x, .) : X \to \mathbb{R}^+$ is lower semicontinuous on for all $x \in X$;
- $(W_3)$ for each $\varepsilon > 0$ there exists $\delta > 0$ such that $q(x, y) \leq \delta$ and $q(x, z) \leq \delta$ imply $d(y, z) \leq \varepsilon$.

**Remark:**

Each metric on a nonempty set $X$ is a $w$-distance on $X$.

**Example 0.2.** [12]. Let $(X, d)$ be a metric space. The function $q : X \times X \to \mathbb{R}^+$ defined by $q(x, y) = c$ for every $x, y \in X$ is a w-distance on $X$, where $c$ is a positive real number. But $q$ is not a metric since $q(x, x) = c \neq 0$ for any $x \in X$.

The following lemma is a useful tool for proving our results.

**Lemma 0.3.** [3]. *Let $(X, d)$ be a metric space, $q$ be a $w$-distance on $X$, $\{x_n\}$ and $\{y_n\}$ be two sequences in $X$, and $x, y, z \in X$.*

(i) *If $\lim_{n\to+\infty} q(x_n, x) = \lim_{n\to+\infty} q(x_n, y) = 0$ then $x = y$. In particular, if $q(z, x) = q(z, y) = 0$ then $x = y$.*

(ii) *If $d(x_n, y_n) \leq \alpha_n$ and $d(x_n, y) \leq \beta_n$ for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, +\infty[$ converging to $0$, then $\{y_n\}$ converges to $y$.*

(iii) *If for each $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $m > n > N_\varepsilon$ implies $q(x_n, x_m) < \varepsilon$, then $\{x_n\}$ is a Cauchy sequence.*

**Definition 0.4.** [12]. A w-distance q on a metric space $(X, d)$ is said to be a ceiling distance of d if and only if

$$q(x, y) \geq d(x, y),$$

for all $x, y \in X$.

**Example 0.5.** [12]. Let $X = \mathbb{R}$ with the metric $d : X \times X \to \mathbb{R}^+$ defined by

$d(x, y) = |x - y|$ for all $x, y \in X$, and let $a, b \geq 1$. Define the function $q : X \times X \to \mathbb{R}^+$ by

$$q(x, y) = \max\{a(y - x), b(x - y)\},$$

for all $x, y \in X$. Ten $q$ is a ceiling distance of $d$.

The following definition was given by Jleli and Samet in [11].

**Definition 0.6.** [11]. Let $\Theta$ be the family of all functions $\theta : ]0, +\infty[ \to ]1, +\infty[$ such that

($\theta_1$) $\theta$ is increasing, i.e., for all $x, y \in ]0, +\infty[$ such that $x < y$, $\theta(x) < \theta(y)$;

($\theta_2$) For each sequence $x_n \in ]0, +\infty[$;

$$\lim_{n\to 0} x_n = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} \theta(x_n) = 1;$$

($\theta_3$) $\theta$ is continuous.

In [13]. Zheng et al. Presented the concept of $\theta - \phi-$contraction on metric spaces as follows.

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**Definition 0.7.** [13] Let $\Phi$ be the family of all functions $\phi$: $[1,+\infty[ \rightarrow [1,+\infty[$, such that

($\phi_1$) $\phi$ is increasing;
($\phi_2$) For each $t > 1$, $lim_{n\rightarrow\infty}\phi^n(t) = 1$;
($\phi_3$) $\phi$ is continuous.

**Lemma 0.8.** [13] *If $\phi \in \Phi$. Then $\phi(1)$=1, and $\phi(t) < t$ for all $t \in ]1,\infty[$.*

**Definition 0.9.** [13]. Let $(X, d)$ be a metric space and $T : X \rightarrow X$ be a mapping.
$T$ is said to be a $\theta - \phi-$contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for any $x, y \in X$,

$$d(Tx,Ty) > 0 \Rightarrow \theta\left[d(Tx,Ty)\right] \leq \varphi\left(\theta\left[N(x,y)\right]\right),$$

where

$$N(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty)\right\}.$$

**Theorem 0.10.** [13]. *Let $(X, d)$ be an complete metric space and let $T : X \rightarrow X$ be an $\theta - \phi$-contraction. Then $T$ has a unique fixed point.*

## 3. Main results

In this paper, using the idea introduce by Wongyat and Sintunavarat [12], we presented the concept of $\theta-$contraction and $\theta - \phi-$contraction on a complete metric space with $w-$distance.

**Definition 0.11.** Let $q$ be a $w-$distance on a metric space $(X, d)$. A mapping $T : X \rightarrow X$ is said to be a $w-$generalized $\theta-$contraction on $(X, d)$ if there exist $\theta \in \Theta$ and $k \in [0, 1[$ such that

$$\theta(q(Tx,Ty)) \leq \left[\theta\left(q(x,y)\right)\right]^r, \tag{0.1}$$

for all $x, y \in X$ for which $Tx \neq Ty$.

**Theorem 0.12.** *Let $(X, d)$ be a complete metric space and $q : X \times X \rightarrow [0,+\infty[$ be a $w-$distance on $X$ and a ceiling distance of $d$. Suppose that $T : X \rightarrow X$ is a $\theta-$contraction. Then, $T$ has an unique fixed point on $X$.*

*Proof.* Let $x_0 \in X$ be an arbitrary point in $X$, define a sequence $\{x_n\}_{n\in\mathbb{N}}$ by

$$x_{n+1} = Tx_n = T^{n+1}x_0,$$

for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $d\left(x_{n_0}, x_{n_0+1}\right) = 0$, then the proof is finished.

We can suppose that $d\left(x_n, x_{n+1}\right) > 0$ for all $n \in \mathbb{N}$.

Since $q$ is a ceiling distance of $d$, we obtain $q\left(x_n, x_{n+1}\right) > 0$ for all $n \in \mathbb{N}$.

Substituting $x = x_{n-1}$ and $y = x_n$, from 0.1, for all $n \in \mathbb{N}$, we have

$$\theta\left[q\left(x_n, x_{n+1}\right)\right] \leq \left[\theta\left(q\left(x_{n-1}, x_n\right)\right)\right]^r, \forall n \in \mathbb{N}. \tag{0.2}$$

Implies that

$$\theta\left[q\left(x_n, x_{n+1}\right)\right] < \theta\left(q\left(x_{n-1}, x_n\right)\right). \tag{0.3}$$

Since $\theta$ is increasing, then $q\left(x_n, x_{n+1}\right) < q\left(x_{n-1}, x_n\right)$. Therefore, $q\left(x_{n+1,}x_n\right)_{n\in\mathbb{N}}$ is monotone strictly decreasing sequence of non negative real numbers. Consequently, there exists $\alpha \geq 0$ such that

$$\lim_{n\to\infty} q\left(x_{n+1,}x_n\right) = \alpha.$$

The inequality 0.2 implies

$$\begin{aligned}\theta\left(q\left(x_n, x_{n+1}\right)\right) &\leq \left[\theta\left(q\left(x_{n-1}, x_n\right)\right)\right]^r \\ &\leq \left[\theta\left(q\left(x_{n-2}, x_{n-1}\right)\right)\right]^{2r} \\ &\leq ... \leq \left[\theta\left(q\left(x_0, x_1\right)\right)\right]^{r^n}.\end{aligned}$$

Since $\theta$ is increasing and continuous function we get

$$q\left(x_n, x_{n+1}\right) < q\left(x_{n-1}, x_n\right). \tag{0.4}$$

Therefore, $q\left(x_{n,}x_{n+1}\right)_{n\in\mathbb{N}}$ is monotone strictly decreasing sequence of non negative real numbers. Consequently, there exists $\alpha \geq 0$ such that

$$\lim_{n\to\infty} q\left(x_{n+1,}x_n\right) \quad = \alpha.$$

Now, we claim that $\alpha = 0$. Arguing by contraction, we assume that $\alpha > 0$. Since $q\left(x_{n,}x_{n+1}\right)_{n\in\mathbb{N}}$ is a non negative decreasing sequence, then we have

$$q\left(x_{n,}x_{n+1}\right) \geq \alpha \quad \forall n \in \mathbb{N}.$$

By property of $\theta$ we get,

$$1<\theta(\alpha)\leq[\theta(q(x_0,x_1))]^{r^n}. \tag{0.5}$$

By letting $n\to\infty$ in inequality 0.5, we obtain

$$1<\theta(\alpha)\leq 1.$$

It is a contradiction. Therefore,

$$\lim_{n\to\infty} q(x_n,x_{n+1})=0. \tag{0.6}$$

Next, we show that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Suppose by contradiction with Lemma 0.3 (iii) that there exist $\varepsilon>0$ and sub-sequences $\{n_{(k)}\}$ and $\{m_{(k)}\}$ of $\{x_n\}_{n\in\mathbb{N}}$ with $n_k>m_k\geq k$ such that $q\left(x_{m_{(k)}},x_{n_{(k)}}\right)\geq\varepsilon$ for all $k\in\mathbb{N}$. Choosing $n_k$ to be the smallest integer exceeding $m_k$ for which $q\left(x_{m_{(k)}},x_{n_{(k)}}\right)\geq\varepsilon$ holds. By the triangular inequality we have,

$$\begin{aligned}\varepsilon &\leq q\left(x_{m_{(k)}},x_{n_{(k)}}\right)\\ &\leq q\left(x_{m_{(k)}},x_{n_{(k)-1}}\right)+q\left(x_{n_{(k)-1}},x_{n_{(k)}}\right)\\ &<\varepsilon+q\left(x_{n_{(k)-1}},x_{n_{(k)}}\right).\end{aligned}$$

Letting $k\to\infty$ the above inequality, we obtain

$$\lim_{k\to\infty} q\left(x_{m_{(k)}},x_{n_{(k)}}\right)=\varepsilon. \tag{0.7}$$

Again by the triangular inequality, for all $n\in\mathbb{N}$, we have the following two inequalities

$$q\left(x_{m_{(k)+1}},x_{n_{(k)+1}}\right)\leq q\left(x_{m_{(k)+1}},x_{m(k)}\right)+q\left(x_{m(k)},x_{n_{(k)}}\right)+q\left(x_{n(k)},x_{n_{(k)+1}}\right), \tag{0.8}$$

$$q\left(x_{m_{(k)}},x_{n_{(k)}}\right)\leq q\left(x_{m_{(k)}},x_{m(k)+1}\right)+q\left(x_{m(k)+1},x_{n_{(k)+1}}\right)+q\left(x_{n(k)+1},x_{n_{(k)}}\right). \tag{0.9}$$

Letting $k\to\infty$ in the above inequalities, we obtain

$$\lim_{k\to\infty} q\left(x_{m_{(k)+1}},x_{n_{(k)+1}}\right)=\varepsilon. \tag{0.10}$$

Now, applying 0.1 with $x=x_{m_{(k)}}$ and $y=x_{n_{(k)}}$, we obtain

$$\theta\left[q\left(x_{m_{(k)+1}},x_{n_{(k)+1}}\right)\right]\leq\left[\theta\left(q\left(x_{m_{(k)}},x_{n_{(k)}}\right)\right)\right]^r. \tag{0.11}$$

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Letting $k\to\infty$ the above inequality and using $(\theta_3)$, we obtain

$$\theta\left(\lim_{k\to\infty} q\left(x_{m_{(k)+1}},x_{n_{(k)+1}}\right)\right)\leq\left[\theta\left(\lim_{k\to\infty} q\left(x_{m_{(k)}},x_{n_{(k)}}\right)\right)\right]^r.$$

Therefore,

$$\theta(\varepsilon) \leq [\theta(\varepsilon)]^r < \theta(\varepsilon).$$

Since $\theta$ is increasing and continuous function we get

$$\varepsilon < \varepsilon,$$

which is a contradiction. Then

$$\lim_{n,m\to\infty} q\left(x_m, x_n\right) = 0. \tag{0.12}$$

By the Lemma 0.3, we can conclude that $\{x_n\}$ is a Cauchy sequence in X. By the completeness of $(X, d)$, there exists $z \in X$ such that

$$\lim_{n\to\infty} d\left(x_n, z\right) = 0. \tag{0.13}$$

Now, we show that $d(Tz, z) = 0$, arguing by contradiction, we assume that

$$d\left(Tz, z\right) > 0 \Rightarrow q\left(Tz, z\right) > 0. \tag{0.14}$$

From 0.12, for each $l > 0$, there is $n_l \in \mathbb{N}$ such that

$$q\left(x_n, x_{n_l}\right) < \frac{1}{l}, \tag{0.15}$$

for all $n_l > l$. Since $q\left(x_{n_l}, .\right)$ is lower semicontinuous and $x_n \to x$ as $n \to \infty$, we get

$$q\left(x_{n_l}, z\right) \leq \liminf_{x\to\infty} q\left(x_{n_l}, x_n\right) \leq \frac{1}{l}, \tag{0.16}$$

implies that

$$\liminf_{n\to\infty} q\left(x_{n_l}, z\right) = 0. \tag{0.17}$$

Now, by triangular inequality we get,

$$q\left(Tx_{n_l}, Tz\right) \leq q\left(Tx_{n_l}, z\right) + q\left(z, Tz\right), \tag{0.18}$$

$$q\left(z, Tz\right) \leq q\left(z, x_{n_l}\right) + q\left(x_{n_l}, Tz\right). \tag{0.19}$$

By letting $n \to \infty$ in inequality $(3.19)$ and $(3.20)$, we obtain

$$q\left(z, Tz\right) \leq \lim_{n\to\infty} q\left(Tx_{n_l}, Tz\right) \leq q\left(z, Tz\right).$$

Therefore,

$$\lim_{n\to\infty} q\left(Tx_{n_l}, Tz\right) = q\left(z, Tz\right). \tag{0.20}$$

Let $A = d(z, Tz) > 0$, from the definition of the limit, there exists $n_2 \in \mathbb{N}$ such that

$$|q(Tx_{n_l}, Tz) - q(z, Tz)| \leq A, \ \forall n \geq n_2,$$

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which implies that

$$q(Tx_{n_l}, Tz) > 0, \ \forall n \geq n_2.$$

Applying 0.1 with $x = z$ and $y = x_{n_l}$, we have

$$\theta(q(Tz, Tx_{n_l})) \leq \theta(q(z, x_{n_l}))^r, \tag{0.21}$$

which implies that

$$\theta(q(Tz, Tx_{n_k})) < \theta(q(z, x_{n_l})).$$

Since $\theta$ is increasing, we get

$$q(Tz, Tx_{n_l}) \leq q(z, x_{n_l}).$$

By letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} q(Tx_{n_l}, Tz) = q(z, Tz) = 0. \tag{0.22}$$

Which is a contradiction, then $d(z, Tz) = q(z, Tz) = 0$, so $Tz = z$.

Uniqueness. Now, suppose that $z, u \in X$ are two fixed points of $T$ such that $u \neq z$. Therefore, we have

$$q(Tz, Tu) = q(z, u) > 0.$$

Applying 0.1 with $x = z$ and $y = u$, we have

$$\begin{aligned} \theta(q(Tu, Tz)) &= \theta(q(z, u)) \\ &\leq [\theta(q(z, u))]^r \\ &< \theta(q(z, u)), \end{aligned}$$

implies

$$q(u, z) < q(u, z),$$

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which is a contradiction. Therefore $u = z$. □

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**Example 0.13.** Let $X = [1, +\infty[$ with the metric $d : X \times X \to [0, +\infty[$ defined by

$$d(x, y) = |x - y|,$$

for all $x, y \in X$. Define a mapping $T : X \to X$ by

$$Tx = \sqrt{x},$$

Suppose that $\theta(t) = e^t$ and $r = \frac{1}{2}$, clearly $\theta \in \Theta$ and $r \in [0,1[$. Also we define a w-distance $q : X \times X \to [0,+\infty[$ by

$$q(x,y) = \max\{x,y\},$$

for all $x, y \in X$. It is easy to see that q is a ceiling distance of d. Now, we will show that $T$ satisfies the condition 0.1.

**Case 1**. If $x \geq y$, then
$q(x,y) = x$, $q(Tx,Ty) = \sqrt{x}$ and $\theta(q(x,y)) = e^x$. Thus

$$[\theta(q(x,y))]^r = e^{\sqrt{x}}$$

and

$$\theta(q(Tx,Ty)) = e^{\sqrt{x}}.$$

We prove that $T$ is a $(\theta)$-contraction mapping. Indeed

$$\theta(q(Tx,Ty) - [\theta(q(x,y))]^r = 0.$$

Therefore,

$$\theta(q(Tx,Ty) \leq [\theta(q(x,y))]^r.$$

**Case 2**. If $x < y$, then
$q(x,y) = y$, $q(Tx,Ty) = \sqrt{y}$ and $\theta(q(x,y)) = e^y$. Thus

$$[\theta(q(x,y))]^r = e^{\sqrt{y}}$$

and

$$\theta(q(Tx,Ty)) = e^{\sqrt{y}}.$$

Therefore,

$$\theta(q(Tx,Ty) \leq [\theta(q(x,y))]^r.$$

Hence, 1 is the unique fixed point of $T$.

**Definition 0.14.** Let $q$ be a w-distance on a metric space $(X,d)$. A mapping $T : X \to X$ is said to be a w-generalized $\theta - \phi-$contraction on $(X,d)$ if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that

$$\theta(q(Tx,Ty)) \leq \phi\left(\theta\left(q(x,y)\right)\right), \tag{0.23}$$

for all $x, y \in X$ for which $Tx \neq Ty$.

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**Theorem 0.15.** *Let $(X,d)$ be a complete metric space and $q : X \times X \to ]0,+\infty[$ be a $w$-distance on $X$ and a ceiling distance of $d$. Suppose that $T : X \to X$ is a $w$-generalized $\theta-\phi-$contraction. Then, $T$ has a unique fixed point on $X$.*

*Proof.* As in the proof of the Theorem 0.12 we can conclude that Let $x_0 \in X$ be an arbitrary point in $X$, define a sequence $\{x_n\}_{n\in\mathbb{N}}$ by

$$x_{n+1} = Tx_n = T^{n+1}x_0,$$

for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0+1}) = 0$, then the proof is finished.

We can suppose that $d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$.

Since $q$ is a ceiling distance of $d$, we obtain $q(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$.

Substituting $x = x_{n-1}$ and $y = x_n$, from (0.23), for all $n \in \mathbb{N}$, we have

$$\theta[q(x_n, x_{n+1})] \leq \phi[\theta(q(x_{n-1}, x_n))], \forall n \in \mathbb{N}. \tag{0.24}$$

By the lemma 0.8, we get

$$\theta[q(x_n, x_{n+1})] < \theta(q(x_{n-1}, x_n)). \tag{0.25}$$

Since $\theta$ is increasing, then $q(x_n, x_{n+1}) < q(x_{n-1}, x_n)$. Therefore, $q(x_{n+1,}x_n)_{n\in\mathbb{N}}$ is monotone strictly decreasing sequence of non negative real numbers. Consequently, there exists $\lambda \geq 0$ such that

$$\lim_{n\to\infty} q(x_{n+1,}x_n) = \lambda.$$

The inequality (0.24) implies

$$\begin{aligned}\theta(q(x_n, x_{n+1})) &\leq \phi(\theta(q(x_{n-1}, x_n))) \\ &\leq \phi^2(\theta(q(x_{n-2}, x_{n-1})) \\ &\leq ... \leq \phi^n(\theta(q(x_0, x_1)).\end{aligned}$$

Now, we claim that $\lambda = 0$. Arguing by contraction, we assume that $\lambda > 0$. Since $q(x_{n,}x_{n+1})_{n\in\mathbb{N}}$ is a non negative decreasing sequence, then we have

$$q(x_{n,}x_{n+1}) \geq \lambda \quad \forall n \in \mathbb{N}.$$

By property of $\theta$ and $\phi$ we get,

$$1 < \theta(\alpha) \leq \phi^n(\theta(q(x_0, x_1)). \tag{0.26}$$

By letting $n \to \infty$ in inequality (0.26), we obtain

$$1 < \theta(\alpha) \leq 1.$$

It is a contradiction. Therefore,

$$\lim_{n\to\infty} q(x_n, x_{n+1}) = 0. \tag{0.27}$$

Next, We shall prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, i.e, $\lim_{n\to\infty} d(x_n, x_m) = 0$, for all $n, m \in \mathbb{N}$. Suppose to the contrary. Then there is an $\varepsilon > 0$ such that for an integer $k$ there exists two sequences $\{n_{(k)}\}$ and $\{m_{(k)}\}$ such that
$q\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \geq \varepsilon$ and $d\left(x_{m_{(k)}}, x_{n_{(k)-1}}\right) < \varepsilon$.

As in the proof of the Theorem 0.12 we can conclude that

$$\lim_{k\to\infty} q\left(x_{m_{(k)}}, x_{n_{(k)}}\right) = \varepsilon. \tag{0.28}$$

and

$$\lim_{k\to\infty} q\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) = \varepsilon. \tag{0.29}$$

Now, applying (0.23) with $x = x_{m_{(k)}}$ and $y = x_{n_{(k)}}$, we obtain

$$\theta\left[q\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)\right] \leq \phi\left[\theta\left(q\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right)\right]. \tag{0.30}$$

Letting $k \to \infty$ the above inequality and using $(\theta_3)$ and $(\phi_3)$, we obtain

$$\theta\left(\lim_{k\to\infty} q\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)\right) \leq \phi\left[\theta\left(\lim_{k\to\infty} q\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right)\right].$$

Therefore,

$$\theta(\varepsilon) \leq \phi[\theta(\varepsilon)] < \theta(\varepsilon).$$

By the Lemma 0.8, we get

$$\varepsilon < \varepsilon,$$

which is a contradiction. Then

$$\lim_{n,m\to\infty} q(x_m, x_n) = 0. \tag{0.31}$$

By the Lemma 0.3, we can conclude that $\{x_n\}$ is a Cauchy sequence in X. By the completeness of $(X, d)$, there exists $z \in X$ such that

$$\lim_{n\to\infty} d(x_n, z) = 0. \tag{0.32}$$

Now, we show that $d(Tz, z) = 0$, arguing by contradiction, we assume that

$$d(Tz, z) > 0 \Rightarrow q(Tz, z) > 0. \tag{0.33}$$

From (0.31), for each $h > 0$, there is $n_h \in \mathbb{N}$ such that

$$q(x_n, x_{n_h}) < \frac{1}{h}, \tag{0.34}$$

for all $n_h > h$. Since $q(x_{n_h}, .)$ is lower semicontinuous and $x_n \to x$ as $n \to \infty$, we get

$$q(x_{n_h}, z) \leq \liminf_{x\to\infty} q(x_{n_h}, x_n) \leq \frac{1}{h}, \tag{0.35}$$

implies that

$$\liminf_{n\to\infty} q(x_{n_h}, z) = 0. \tag{0.36}$$

As in the proof of the Theorem 0.12 we can conclude that ,

$$\lim_{n\to\infty} q(Tx_{n_h}, Tz) = q(z, Tz). \tag{0.37}$$

Let $B = d(z, Tz) > 0$, from the definition of the limit, there exists $n_1 \in \mathbb{N}$ such that

$$|q(Tx_{n_h}, Tz) - q(z, Tz)| \leq B, \ \forall n \geq n_1,$$

which implies that

$$q(Tx_{n_h}, Tz) > 0, \ \forall n \geq n_1.$$

Applying (0.23) with $x = z$ and $y = x_{n_h}$, we have

$$\theta(q(Tz, Tx_{n_h})) \leq \phi[\theta(q(z, x_{n_h}))], \tag{0.38}$$

By Lemma 0.8, we get

$$\theta(q(Tz, Tx_{n_k})) < \theta(q(z, x_{n_h})).$$

Since $\theta$ is increasing, we get

$$q(Tz, Tx_{n_h}) \leq q(z, x_{n_h}).$$

By letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} q(Tx_{n_h}, Tz) = q(z, Tz) = 0. \tag{0.39}$$

Which is a contradiction, then $d(z, Tz) = q(z, Tz) = 0$, so $Tz = z$.

Uniqueness. Now, suppose that $z, u \in X$ are two fixed points of $T$ such that $u \neq z$. Therefore, we have

$$q(Tz, Tu) = q(z, u) > 0.$$

Applying (0.2) with $x = z$ and $y = u$, we have

$$\begin{aligned}\theta\left(q\left(Tu, Tz\right)\right) &= \theta\left(q\left(z, u\right)\right)\\ &\leq \phi\left[\theta\left(q\left(z, u\right)\right)\right]\\ &< \theta\left(q\left(z, u\right)\right),\end{aligned}$$

implies

$$q\left(u, z\right) < q\left(u, z\right),$$

which is a contradiction. Therefore $u = z$. $\square$

**Example 0.16.** Let $X = [0, +\infty[$ with the metric $d : X \times X \to [0, +\infty[$ defined by

$$d(x, y) = |x - y|,$$

for all $x, y \in X$. Define a mapping $T : X \to X$ by

$$Tx = \frac{x}{4},$$

Suppose that $\theta(t) = \sqrt{t} + 1$ and $\phi(t) = \frac{t+1}{2}$, clearly $\theta \in \Theta$ and $\phi \in \Phi$. Also we define a w-distance $q : X \times X \to [0, +\infty[$ by

$$q(x, y) = \max\left\{x, y\right\},$$

for all $x, y \in X$. It is easy to see that q is a ceiling distance of d. Now, we will show that $T$ satisfies the condition (0.23).

**Case 1**. If $x \geq y$, then

$q(x, y) = x$, $q(Tx, Ty) = \frac{x}{4}$ and $\theta(q(x, y)) = \sqrt{x} + 1$. Thus

$$\phi\left[\theta(q(x, y))\right] = \frac{\sqrt{x} + 1 + 1}{2} = \frac{\sqrt{x}}{2} + 1$$

and

$$\theta(q(Tx, Ty)) = \frac{\sqrt{x}}{2} + 1.$$

We prove that $T$ is a w-generalized $\theta - \phi-$contraction. Indeed

$$\theta(q(Tx, Ty) - \phi\left[\theta(q(x, y))\right] = 0.$$

Therefore,

$$\theta(q(Tx, Ty) \leq \phi\left[\theta(q(x, y))\right].$$

**Case 2**. If $x < y$, then $q(x, y) = y$, $q(Tx, Ty) = \frac{y}{4}$ and $\theta(q(x, y)) = \sqrt{y} + 1$. Thus

$$\phi\left[\theta(q(x, y))\right] = \frac{\sqrt{y} + 1 + 1}{2} = \frac{\sqrt{y}}{2} + 1$$

and

$$\theta(q(Tx,Ty)) = \frac{\sqrt{y}}{2} + 1.$$

We prove that $T$ is a w-generalized $\theta - \phi-$contraction. Indeed

$$\theta(q(Tx,Ty) - \phi\left[\theta(q(x,y))\right] = 0.$$

Therefore,

$$\theta(q(Tx,Ty) \leq \phi\left[\theta(q(x,y))\right].$$

Hence, 0 is the unique fixed point of $T$.

**Example 0.17.** Let X be the set defined by

$$X = \{\lambda_n : n \in \mathbb{N}^*\},$$

where

$$\lambda_n = \frac{(n)(n+1)}{2}$$

. Let the metric $d : X\times \to [0,+\infty[$ defined by

$$d(x,y) = |x-y|$$

for all $x, y \in X$. Define a mapping $T : X \to X$ by

$$T(\lambda_n) = \begin{cases} 1 \ if \ n = 1 \\ n(n-1) \ if \ n \geq 2. \end{cases}$$

Clearly, the Banach contraction is not satisfied. In fact, we can check easily that

$$\lim_{n\to\infty} \frac{d(T(\lambda_n), T(\lambda_1))}{d(\lambda_n, \lambda_1)} = \lim_{n\to\infty} \frac{n^2-n-2}{n^2+n-2} = 1.$$

Now, let the function $\theta(t) = e^t$ and

$$\phi(t) = \begin{cases} 1 \ if \ 1 \leq t \leq 2 \\ t-1 \ if \ t \geq 2. \end{cases}$$

Obviously, $\theta \in \Theta$ and $\phi \in \Phi$. Also we define a w-distance $q : X\times \to [0,+\infty[$ by

$$q(x,y) = \max\{x,y\}$$

for all $x, y \in X$. It is easy to see that q is a ceiling distance of d. Now, we will show that $T$ satisfies the condition (0.2).

**Case 1**. $m = 1$ and $n \geq 2$ . In this case, we have

$$q(\lambda_n, \lambda_1) = \frac{n(n+1)}{2}, q(T(\lambda_n), T(\lambda_1)) = \frac{n(n-1)}{2}$$

and

$$\theta(q(T\lambda_n, T\lambda_1)) = e^{\frac{n(n-1)}{2}}.$$

and

$$\phi\left[\theta(q(\lambda_n), \lambda_1))\right] = e^{\frac{n(n+1)}{2}} - 1,$$

On the other hand

$$\begin{aligned}\theta(q(T\lambda_n, T\lambda_1)) - \phi\left[\theta(q(\lambda_n), \lambda_1))\right] &= e^{\frac{n(n-1)}{2}} - e^{\frac{n(n+1)}{2}} + 1 \\ &= e^{\frac{n(n-1)}{2}}\left[1 - \frac{e^{\frac{n(n+1)}{2}}}{e^{\frac{n(n-1)}{2}}}\right] + 1 \\ &= e^{\frac{n(n-1)}{2}}\left[1 - e^{n}\right] + 1 \\ &\leq 0.\end{aligned}$$

Therefore,

$$\theta(q(T\lambda_n, T\lambda_1)) \leq \phi\left[\theta(q(\lambda_n), \lambda_1))\right].$$

Case 2. $m > n > 1$. In this case, we have

$$q(\lambda_n, \lambda_m) = \frac{m(m+1)}{2}, q(T(\lambda_n), T(\lambda_m)) = \frac{m(m-1)}{2}$$

and

$$\theta(q(T(\lambda_n), T(\lambda_m))) = e^{\frac{m(m-1)}{2}}.$$

and

$$\phi\left[\theta(q(\lambda_n, \lambda_m))\right] = e^{\frac{m(m+1)}{2}} - 1,$$

On the other hand

$$\begin{aligned}\theta(q(\lambda_n, \lambda_m)) - \phi\left[\theta(q(\lambda_n, \lambda_m))\right] &= e^{\frac{m(m-1)}{2}} - e^{\frac{m(m+1)}{2}} + 1 \\ &\leq 0.\end{aligned}$$

Therefore,

$$\theta(q(\lambda_n, \lambda_m)) \leq \phi\left[\theta(q(\lambda_n, \lambda_m))\right].$$

Thus, the inequality (0.23) is satisfied implies that $T$ has a unique fixed point. In this example $\lambda_1$ is the unique fixed point of $T$.

Taking $q = d$ in Theorems 0.12 and 0.15, we obtain the following result.

**Corollary 0.18.** *Let $(X, d)$ be a complete metric space and $T : X \to X$ be a $\theta-$contraction . Then $T$ has a unique fixed point.*

**Corollary 0.19.** *Let $(X, d)$ be a complete metric space and $T : X \to X$ be a $\theta-\phi-$contraction. Then $T$ has a unique fixed point.*

## 1. Application to nonlinear integral equations

In this section, we endeavor to apply the Theorems 0.12 and 0.15 to prove the existence and uniqueness of the integral equation of Fredholm type:

$$x(t) = \lambda \int_a^b K(t, s, x(s))ds, \tag{1.1}$$

where $a, b \in \mathbb{R}$, $x \in C([a, b], \mathbb{R})$ and $K : [a, b]^2 \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

**Theorem 1.1.** *Consider the nonlinear integral equation problem:* (4.1) *and assume that the kernel function $K$ satisfies the condition $|K(t, r, x(r))|+|K(t, r, y(r))| \leq k(|x(t) + y(t)|)$ for all $t, r \in [a, b]$, $k \in ]0, 1[$ and $x, y \in \mathbb{R}$. Then the equation* (4.1) *has a unique solution $x \in C([a, b]$ for some constant $\lambda$ depending on the constants $k$.*

*Proof.* Let $X = C([a, b]$ and $T : X \to X$ defined by

$$T(x)(t) = \lambda \int_a^b K(t, s, x(s))ds,$$

for all $x \in X$. Clearly, $X$ with the metric $d : X \times X \to [0, +\infty[$ given by

$$d(x, y) = \sup_{t\in[a,b]} |x(t) - y(t)|,$$

for all $x, y \in X$, is a complete metric space. Next, define the function $q : X \times X \to [0, +\infty[$ by

$$q(x, y) = \sup_{t\in[a,b]} |x(t)| + |y(t)|,$$

for all $x, y \in X$.

Clearly, $q$ is a w-distance on $X$ and a ceiling distance of d. We will find the condition on $\lambda$ under which the operator has a unique fixed point which will the solution of the integral equation (4.1). Assume that, $x, y \in X$ and $t, s \in [a, b]$. Then we get

$$\begin{aligned} |Tx(t)| + |Ty(t)| &= |\lambda| \left( |\int_a^b K(t, s, x(s))ds| + |\int_a^b K(t, s, y(s))ds| \right) \\ &\leq |\lambda| \int_a^b |K(t, s, x(s))ds| + |\lambda| \int_a^b |K(t, s, y(s))ds| \\ &\leq k|\lambda| \int_a^b (|x(s)| + |y(s)|)\, ds, \end{aligned}$$

which implies that

$$\begin{aligned}\sup_{t\in[a,b]}(|Tx(t)|+|Ty(t)|) &= \sup_{t\in[a,b]}\left(|\lambda|\left|\left(\int_a^b K(t,x(s))ds\right|+\left|\int_a^b K(t,y(s))ds\right|\right)\right)\\ &\le \sup_{t\in[a,b]} \le |\lambda|\int_a^b |K(t,s,x(s))ds| + |\lambda|\int_a^b |K(t,s,y(s))ds|\\ &\le k|\lambda|\int_a^b \left(\left(\sup_{s\in[a,b]}|x(s)|+|y(s)|\right)\right)ds.\end{aligned}$$

Since by the definition of the w-distance on $X$ and a ceiling distance of d, we have $q(Tx,Ty)>0$ and $q(x,y)>0$ for any $x\neq y$, then we can take natural exponential sides and get

$$\begin{aligned}e^{[q(Tx,Ty)]} &= e^{\left[|\lambda|\max_{t\in[a,b]}\int_a^b|K(t,r,x(r))+K(t,r,y(r))dr|\right]}\\ &\le e^{\left[k|\lambda|\int_a^b\left(\left(\max_{r\in[a,b]}|x(r)+y(r)|\right)dr\right)\right]}\\ &= \left[e^{\left[\int_a^b\left(\left(\max_{r\in[a,b]}|x(r)+y(r)|\right)dr\right)\right]}\right]^{k|\lambda|}.\end{aligned}$$

provided that $k|\lambda|<1$, which implies that

$$e^{[q(Tx,Ty)]} \le \left[e^{\left[\int_a^b\left(\left(\max_{r\in[a,b]}|x(r)+y(r)|\right)dr\right)\right]}\right]^{|\lambda|k}.$$

Hence

$$\theta\left(d(Tx,Ty)\right)\le\phi\left[\theta\left(d(x,y)\right)\right], \tag{1.2}$$

for all x,y $\in X$ with $\theta(t)=e^t$, $\phi(t)=t^r$ and $r=|\lambda|k$. It follows that $T$ satisfies the condition (0.1) and (0.23). Therefore there exists a unique solution of the nonlinear Fredholm inequality (1.1). □

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