JURNAL MATEMATIKA, STATISTIKA & KOMPUTASI E-ISSN: 2614-8811
https://journal.unhas.ac.id/index.php/jmsk/index
Vol. 18, No. 3, 407-422, May, 2022
DOI: 10.20956/j.v18i3.20459

Some properties of K-operator Frame in Hilbert C^*-modules

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Abstract

In this paper, we present some properties of K-operator Frame in Hilbert C^*-modules. Topics that will be discussed include: K-operator Frame and Dual K-operator frame in Hilbert C^*-modules. We will also study K-operator Frame in two Hilbert C^*-modules with different C^*-algebras.

Keywords: K-operator Frame, Dual K-operator frame, C^*-algebra, Hilbert C^*-modules, Tensor Product.

1. INTRODUCTION AND PRELIMINARIES

Frame theory is recently an active research area in mathematics, computer science, and engineering with many exciting applications in a variety of different fields. They are generalizations of bases in Hilbert spaces. Frames for Hilbert spaces were first introduced in 1952 by Duffin and Schaefer [6] for study of nonharmonic Fourier series. They were reintroduced and developed in 1986 by Daubechies, Grossmann and Meyer [4], and popularized from then on.

Hilbert C^*-modules is a generalization of Hilbert spaces by allowing the inner product to take values in a C^*-algebra rather than in the field of complex numbers. For more details about Hilbert C^*-modules frames, see [10, 11, 12, 13, 15, 16, 17].

In the following, we recall some definitions and results that will be used to prove our mains results.

For a C^*-algebra $\mathcal{A}$, an element $a \in \mathcal{A}$ is positive ($a \geq 0$) if $a = a^*$ and $sp(a) \subset \mathbf{R}^+$. $\mathcal{A}^+$ denotes the set of positive elements of $\mathcal{A}$ For more details see [3, 5].

Definition 0.1. [9]. Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{H}$ be a left $\mathcal{A}$-module, such that the linear structures of $\mathcal{A}$ and $\mathcal{H}$ are compatible. $\mathcal{H}$ is a pre-Hilbert $\mathcal{A}$-module if $\mathcal{H}$ is

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equipped with an $\mathcal{A}$-valued inner product $\langle ., . \rangle_{\mathcal{A}} : \mathcal{H} \times \mathcal{H} \to \mathcal{A}$, such that is sesquilinear, positive definite and respects the module action. In the other words,

(i) $\langle x, x \rangle_{\mathcal{A}} \geq 0$ for all $x \in \mathcal{H}$ and $\langle x, x \rangle_{\mathcal{A}} = 0$ if and only if $x = 0$.

(ii) $\langle ax + y, z \rangle_{\mathcal{A}} = a\langle x, y \rangle_{\mathcal{A}} + \langle y, z \rangle_{\mathcal{A}}$ for all $a \in \mathcal{A}$ and $x, y, z \in \mathcal{H}$.

(iii) $\langle x, y \rangle_{\mathcal{A}} = \langle y, x \rangle_{\mathcal{A}}^*$ for all $x, y \in \mathcal{H}$.

For $x \in \mathcal{H}$, we define $||x|| = ||\langle x, x \rangle_{\mathcal{A}}||^{\frac{1}{2}}$. If $\mathcal{H}$ is complete with $||.||$, it is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over $\mathcal{A}$. For every a in C^*-algebra $\mathcal{A}$, we have $|a| = (a^*a)^{\frac{1}{2}}$ and the $\mathcal{A}$-valued norm on $\mathcal{H}$ is defined by $|x| = \langle x, x \rangle_{\mathcal{A}}^{\frac{1}{2}}$ for $x \in \mathcal{H}$.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules, A map $T : \mathcal{H} \to \mathcal{K}$ is said to be adjointable if there exists a map $T^* : \mathcal{K} \to \mathcal{H}$ such that $\langle Tx, y \rangle_{\mathcal{A}} = \langle x, T^*y \rangle_{\mathcal{A}}$ for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$.

We also reserve the notation $End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{K})$ for the set of all adjointable operators from $\mathcal{H}$ to $\mathcal{K}$ and $End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{H})$ is abbreviated to $End_{\mathcal{A}}^*(\mathcal{H})$.

Let I and J be countable index sets.

Definition 0.2. [7] A family of adjointable operators $\{T_i\}_{i \in I}$ on a Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$ is said to be an operator frames for $End_{\mathcal{A}}^*(\mathcal{H})$, if there exist two positive constants $A, B > 0$ such that

$$A\langle x, x \rangle_{\mathcal{A}} \leq \sum_{i \in I} \langle T_i x, T_i x \rangle \leq B\langle x, x \rangle_{\mathcal{A}}, \quad x \in \mathcal{H}. \tag{0.1}$$

The numbers A and B are called lower and upper bound of the operator frames, respectively. If $A = B = \lambda$, the operator frame is λ-tight.

If $A = B = 1$, it is called a normalized tight operator frames or a Parseval operator frames.

If only upper inequality of (0.1) hold, then $\{T_i\}_{i \in I}$ is called an operator Bessel sequence for $End_{\mathcal{A}}^*(\mathcal{H})$.

Definition 0.3. [14]. Let $K \in End_{\mathcal{A}}^*(\mathcal{H})$. A family of adjointable operators $\{\Lambda_i\}_{i \in I}$ on a Hilbert $\mathcal{A}$-module $\mathcal{H}$ is said to be a K-operator frame for $\mathcal{H}$, if there exists positive constants $A, B > 0$ such that

$$A\langle K^*x, K^*x \rangle_{\mathcal{A}} \leq \sum_{i \in I} \langle \Lambda_i x, \Lambda_i x \rangle_{\mathcal{A}} \leq B\langle x, x \rangle_{\mathcal{A}}, \forall x \in \mathcal{H}. \tag{0.2}$$

The numbers A and B are called lower and upper bounds of the K-operator frame, respectively. If

$$A\langle K^*x, K^*x\rangle_{\mathcal{A}} = \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}},$$

the K-operator frame is A-tight. If $A = 1$, it is called a normalized tight K-operator frame or a Parseval K-operator frame.

If the sum in the middle of (0.2) is convergent in norm, the operator frame is called standard.

Throughout the paper, series like (0.2) are assumed to be convergent in the norm sense.

Let $\{\Lambda_i\}_{i\in I}$ be a K-operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$. Define an operator

$$R : \mathcal{H} \to l^2(\mathcal{H}) \ \ by \ \ Rx = \{\Lambda_i x\}_{i\in I}, \forall x \in \mathcal{H}.$$

The operator R is called the analysis operator of the K-operator frame $\{\Lambda_i\}_{i\in I}$.

The adjoint of the analysis operator R,

$$R^*(\{x_i\}_{i\in I}) : l^2(\mathcal{H}) \to \mathcal{H}$$

is defined by

$$R^*(\{x_i\}_{i\in I}) = \sum_{i\in I} \Lambda_i^* x_i, \forall \{x_i\}_{i\in I} \in l^2(\mathcal{H}).$$

The operator R^* is called the synthesis operator of the K-operator frame $\{\Lambda_i\}_{i\in I}$.

By composing R and R^*, the frame operator $S_T : \mathcal{H} \to \mathcal{H}$ for the K-operator frame is given by

$$S_T(x) = R^*Rx = \sum_{i\in I} \Lambda_i^*\Lambda_i x.$$

Lemma 0.4. [8]. *If $Q \in End^*_{\mathcal{A}}(\mathcal{H})$ is an invertible $\mathcal{A}$-linear map then for all $z \in \mathcal{H}\otimes\mathcal{K}$ we have*

$$\|Q^{*-1}\|^{-1}.|z| \leq |(Q^* \otimes I)z| \leq \|Q\|.|z|.$$

Lemma 0.5. [1]. *If $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is a $*$-homomorphism between $\mathcal{C}^*$-algebras, then φ is increasing, that is, if $a \leq b$, then $\varphi(a) \leq \varphi(b)$.*

Lemma 0.6. [18]. *Let $\mathcal{H}$ be Hilbert $\mathcal{A}$ -module over a* C**-algebra $\mathcal{A}$. Let $T, S \in$* $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$. *If* $\mathrm{Ran}(S)$ *is closed, then the following statements are equivalent:*

(i) $\mathrm{Ran}(T) \subseteq \mathrm{Ran}(S)$.*description*

(ii) $TT^* \leq \lambda^2 SS^*$ *for some* $\lambda > 0$.

(iii) *There exists* $Q \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$ *such that* $T = SQ$.

Proposition 0.7. [2] *Let* $\mathcal{A}$ *be a* C^**-algebra,* $\mathcal{V}$ *and* $\mathcal{W}$ *Hilbert A-modules, and* $T \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{V}, \mathcal{W})$ *The following statements are mutually equivalent:*

(1) T *is surjective.*

(2) T^* *is bounded below with respect to the norm, i.e., there is* $m > 0$ *such that* $\|T^*x\| \geq m\|x\|$ *for all* $x \in \mathcal{V}$

(3) T^* *is bounded below with respect to the inner product, i.e., there is* $m' > 0$ *such that* $\langle T^*x, T^*x\rangle \geq m'\langle x, x\rangle$ *for all* $x \in \mathcal{V}$.

In this article, we study some new properties of K-operator frame in Hilbert C^*-modules. Also, we define and study the dual K-operator frame. Our theorems extend, generalize and improve many existing results.

2. MAIN RESULTS

The following theorem give a characterization for a K-operator frame for $End^*_A(\mathcal{H})$.

Theorem 0.8. *For an operator Bessel sequence* $\{\Lambda_i\}_{i\in I} \subset End^*_A(\mathcal{H})$*, the following statements are equivalent:*

1. $\{\Lambda_i\}_{i\in I}$ *is a* K*-operator frame for* $\mathrm{End}^*_A(\mathcal{H})$.
2. *There exists* $A > 0$ *such that* $S \geq AKK^*$*, where* S *is the frame operator for* $\{\Lambda_i\}_{i\in I}$.
3. $K = S^{\frac{1}{2}}Q$*, for some* $Q \in End^*_{\mathcal{A}}(\mathcal{H})$*, where* $S^{\frac{1}{2}}$ *is the square root of the operator* S.

Proof. 1. $\Rightarrow$ 2. Note that $\{\Lambda_i\}_{i\in I}$ is a K-operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$ with frame bounds A and B and frame operator S if and only if

$$A\,\langle K^*x, K^*x\rangle \leq \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle \leq B\langle x, x\rangle, \forall x \in \mathcal{H}.$$

Thus, we have

$$\langle AKK^*x, x\rangle \leq \langle Sx, x\rangle \leq \langle Bx, x\rangle, \forall x \in \mathcal{H}.$$

Hence $S \geq AKK^*$.

2. $\Rightarrow$ 3. Suppose that there exists $A > 0$ such that $AKK^* \leq S$. This give $AKK^* \leq S^{\frac{1}{2}}S^{\frac{1}{2}*}$. Then by Lemma 0.6, $K = S^{\frac{1}{2}}Q$, for some $Q \in End^*_{\mathcal{A}}(\mathcal{H})$.

3. $\Rightarrow$ 1. Let $K = S^{\frac{1}{2}}Q$, for some $Q \in End^*_{\mathcal{A}}(\mathcal{H})$. Then by Lemma 0.6, there exists $A > 0$ such that $AKK^* \leq S^{\frac{1}{2}}S^{\frac{1}{2}*}$. This give $AKK^* \leq S$. Hence $\{\Lambda_i\}_{i\in I}$ is a K-operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$. □

Theorem 0.9. *A family $\{\Lambda_i\}_{i\in I} \subset End^*_{\mathcal{A}}(\mathcal{H})$ is a K-operator frame if and only if $Ran(K) \subset Ran(R^*)$, where R is the analysis operator.*

Proof. Let $\{\Lambda_i\}_{i\in I}$ be a K-operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$. Then there exists $A > 0$ such that $S \geq AKK^*$, where S is the frame operator for $\{\Lambda_i\}_{i\in I}$.

Since $S = R^*R$ then $R^*R \geqslant AKK^*$. Therefore by Lemma 0.6 $\mathrm{Ran}(K) \subseteq \mathrm{Ran}(R^*)$.

Conversely, suppose that $\mathrm{Ran}(K) \subseteq \mathrm{Ran}(R^*)$. Then $KK^* \leq \lambda^2 R^*R$. Thus $KK^* \leq \lambda^2 S$. Therefore by Theorem 0.8 $\{\Lambda_i\}_{i\in I}$ is a K-operator frame for $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$ □

Next, we show that K-operator frame for $\mathcal{H}$ is invariant under the composition of the operators.

Theorem 0.10. *Let $K \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$ with a dense range and $\{\Lambda_i\}_{i\in I}$ be a K-operator frame for $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$. If $Q \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$ has closed range with $QK = KQ$, then $\{\Lambda_i Q^*\}_{i\in I}$ is K-operator frame for $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$ if and only if Q is surjective.*

Proof. We have

$$A\langle K^*Q^*x, K^*Q^*x\rangle = A\langle Q^*K^*x, Q^*K^*x\rangle$$

Suppose that Q is surjective. Then by Proposition 0.7 there exists $m > 0$ such that

$$Am\langle K^*x, K^*x\rangle \leq \; A\langle Q^*K^*x, Q^*K^*x\rangle \leq \sum_{i\in I}\langle \Lambda_i Q^*x, \Lambda_i Q^*x\rangle, x \in \mathcal{H}.$$

Also, we have

$$\sum_{i\in I}\langle \Lambda_i Q^*x, \Lambda_i Q^*x\rangle \leq B\langle Q^*x, Q^*x\rangle \leq B\|Q\|^2\langle x, x\rangle.$$

Hence, $\{\Lambda_i Q^*\}_{i\in I}$ is a K-operator frame for $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$.

Conversely, suppose $\{\Lambda_i Q^*\}_{i\in I}$ is K-operator frame for $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$ with frame bounds A and B Then for any $x \in \mathcal{H}$, we have

$$A\langle K^*x, K^*x\rangle \leq \sum_{i\in I}\langle \Lambda_i Q^*x, \Lambda_i Q^*x\rangle \leq B\langle x, x\rangle \quad (2.2)$$

Since, $Ran(K)$ is dense in $\mathcal{H}$, K^* is injective. Thus, using (2.2), Q^* is injective. □

Theorem 0.11. *Let* $K \in \mathrm{End}_{\mathcal{A}}^{*}(\mathcal{H})$ *and* $\{\Lambda_i\}_{i\in I}$ *be a* K*-operator frame for* $\mathrm{End}_{\mathcal{A}}^{*}(\mathcal{H})$*. If* $Q \in \mathrm{End}_{\mathcal{A}}^{*}(\mathcal{H})$ *be an isometry with* $K^*Q = QK^*$*, then* $\{\Lambda_i Q\}_{i\in I}$ *is* K*-operator frame for* $\mathrm{End}_{\mathcal{A}}^{*}(\mathcal{H})$*.*

Proof. Suppose $\{\Lambda_i\}_{i\in I}$ is K-operator frame for $\mathrm{End}_{\mathcal{A}}^{*}(\mathcal{H})$. Then, for each $x \in \mathcal{H}$, we have

$$\sum_{i\in I} \langle \Lambda_i Qx, \Lambda_i Qx\rangle \geq A\,\langle K^*Qx, K^*Qx\rangle$$

$$= A\,\langle QK^*x, QK^*x\rangle$$

$$= A\,\langle K^*x, K^*x\rangle$$

Also,

$$\sum_{i\in I} \langle \Lambda_i Qx, \Lambda_i Qx\rangle \leq B\|Q\|^2\langle x, x\rangle$$

Hence $\{\Lambda_i Q\}_{i\in I}$ is a K-operator frame for $\mathrm{End}_{\mathcal{A}}^{*}(\mathcal{H})$. □

Theorem 0.12. *Let* $\{\Lambda_i\}_{i\in I}$ *and* $\{R_i\}_{i\in I}$ *be* K*-operators frame for* $\mathcal{H}$ *with frame operators* S_1 *and* S_2 *respectively. Then* $K = S_1^{1/2}P + S_2^{1/2}Q$ *for some* $P, Q \in \mathrm{End}_{\mathcal{A}}^{*}(\mathcal{H})$*.*

Proof. Let $\{\Lambda_i\}_{i\in I}$ and $\{R_i\}_{i\in I}$ be K-operators frame for $\mathcal{H}$ with frame operators S_1 and S_2 respectively. Then by Theorem 0.8, there exist $\alpha_1, \alpha_2 > 0$ such that $S_1 \geq \alpha_1 KK^*$ and $S_2 \geq \alpha_2 KK^*$. Therefore, by Lemma 0.6, we get $Ran(K) \subset Ran\left(S_1^{1/2}\right)$ and $Ran(K) \subset Ran\left(S_2^{1/2}\right)$. Hence $Ran(K) \subset Ran\left(S_1^{1/2}\right) + Ran\left(S_2^{1/2}\right)$. Thus, we obtain $K = S_1^{1/2}P + S_2^{1/2}Q$ for some $P, Q \in \mathrm{End}_{\mathcal{A}}^{*}(\mathcal{H})$. □

Theorem 0.13. *Let* $\{\Lambda_i\}_{i\in I}$ *be a* K*-operator frame for* $\mathcal{H}$ *with the frame operator* S *and let* Q *be a positive operator such that* $SQ = QS$*. Then* $\{\Lambda_i + \Lambda_i Q\}_{i\in I}$ *is a* K*-operator frame for* $\mathcal{H}$*. Moreover, for any natural number* $n, \{\Lambda_i + \Lambda_i Q^n\}_{i\in I}$ *is a* K*-operator frame for* $\mathcal{H}$

Proof. Let $\{\Lambda_i\}_{i\in I}$ be a K-operator frame for $\mathcal{H}$ with the frame operator S. Then, there exist $\lambda > 0$ such that $S \geq \lambda KK^*$. The frame operator for $\{\Lambda_i + \Lambda_i Q\}_{i\in I}$ is given by

$$\sum_{i\in I} (\Lambda_i + \Lambda_i Q)^* (\Lambda_i + \Lambda_i Q)(x) = (\mathcal{I} + Q)^* S(\mathcal{I} + Q)(x)$$

Since $(\mathcal{I} + Q)^* S(\mathcal{I} + Q) \geq S \geq \lambda KK^*$, then $\{\Lambda_i + \Lambda_i Q\}_{i\in I}$ is a K-operator frame for H.

Similarly, for any natural number n, $\{\Lambda_i + \Lambda_i Q^n\}_{i\in I}$ is a K-operator frame for $\mathcal{H}$. □

In the following, we study K-operator frame in tensor products of Hilbert C^*-modules.

Theorem 0.14. *Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert C^*-modules over unital C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $\{\Lambda_i\}_{i\in I} \subset End_{\mathcal{A}}^*(\mathcal{H})$ be a K_1-operator frame for $\mathcal{H}$ and $\{\Gamma_j\}_{j\in J} \subset End_{\mathcal{B}}^*(\mathcal{K})$ be a K_2-operator frame for $\mathcal{K}$ with frame operators S_Λ and S_Γ and operator frame bounds (A,B) and (C,D) respectively. Then $\{\Lambda_i \otimes \Gamma_j\}_{i\in I, j\in J}$ is a $K_1 \otimes K_2$-operator frame for Hibert $\mathcal{A}\otimes\mathcal{B}$-module $\mathcal{H}\otimes\mathcal{K}$ with frame operator $S_\Lambda \otimes S_\Gamma$ and lower and upper operator frame bounds AC and BD, respectively.*

Proof. By the definition of K_1-operator frame $\{\Lambda_i\}_{i\in I}$ and K_2-operator frame $\{\Gamma_j\}_{j\in J}$ we have

$$A\langle K_1^* x, K_1^* x\rangle_{\mathcal{A}} \leq \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}} \leq B\langle x, x\rangle_{\mathcal{A}}, \forall x \in \mathcal{H}.$$

$$C\langle K_2^* y, K_2^* y\rangle_{\mathcal{B}} \leq \sum_{j\in J}\langle \Gamma_j y, \Gamma_j y\rangle_{\mathcal{B}} \leq D\langle y, y\rangle_{\mathcal{B}}, \forall y \in \mathcal{K}.$$

Therefore

$$\begin{aligned}
&(A\langle K_1^* x, K_1^* x\rangle_{\mathcal{A}}) \otimes (C\langle K_2^* y, K_2^* y\rangle_{\mathcal{B}}) \\
&\leq \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}} \otimes \sum_{j\in J}\langle \Gamma_j y, \Gamma_j y\rangle_{\mathcal{B}} \\
&\leq (B\langle x, x\rangle_{\mathcal{A}}) \otimes (D\langle y, y\rangle_{\mathcal{B}}), \forall x \in \mathcal{H}, \forall y \in \mathcal{K}.
\end{aligned}$$

Then

$$\begin{aligned}
&AC(\langle K_1^* x, K_1^* x\rangle_{\mathcal{A}} \otimes \langle K_2^* y, K_2^* y\rangle_{\mathcal{B}}) \\
&\leq \sum_{i\in I, j\in J}\langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}} \otimes \langle \Gamma_j y, \Gamma_j y\rangle_{\mathcal{B}} \\
&\leq BD(\langle x, x\rangle_{\mathcal{A}} \otimes \langle y, y\rangle_{\mathcal{B}}), \forall x \in \mathcal{H}, \forall y \in \mathcal{K}.
\end{aligned}$$

Consequently we have

$$\begin{aligned}
&AC\langle K_1^* x \otimes K_2^* y, K_1^* x \otimes K_2^* y\rangle_{\mathcal{A}\otimes\mathcal{B}} \\
&\leq \sum_{i\in I, j\in J}\langle \Lambda_i x \otimes \Gamma_j y, \Lambda_i x \otimes \Gamma_j y\rangle_{\mathcal{A}\otimes\mathcal{B}} \\
&\leq BD\langle x \otimes y, x \otimes y\rangle_{\mathcal{A}\otimes\mathcal{B}}, \forall x \in \mathcal{H}, \forall y \in \mathcal{K}.
\end{aligned}$$

Then for all $x \otimes y$ in $\mathcal{H} \otimes \mathcal{K}$ we have

$$\begin{aligned}
&AC\langle (K_1 \otimes K_2)^*(x \otimes y), (K_1 \otimes K_2)^*(x \otimes y)\rangle_{\mathcal{A}\otimes\mathcal{B}} \\
&\leq \sum_{i\in I, j\in J} \langle (\Lambda_i \otimes \Gamma_j)(x \otimes y), (\Lambda_i \otimes \Gamma_j)(x \otimes y)\rangle_{\mathcal{A}\otimes\mathcal{B}} \\
&\leq BD\langle x \otimes y, x \otimes y\rangle_{\mathcal{A}\otimes\mathcal{B}}.
\end{aligned}$$

The last inequality is satisfied for every finite sum of elements in $\mathcal{H} \otimes_{alg} \mathcal{K}$ and then it's satisfied for all $z \in \mathcal{H} \otimes \mathcal{K}$. It shows that $\{\Lambda_i \otimes \Gamma_j\}_{i\in I, j\in J}$ is a $K_1 \otimes K_2$-operator frame for Hilbert $\mathcal{A} \otimes \mathcal{B}$-module $\mathcal{H} \otimes \mathcal{K}$ with lower and upper operator frame bounds AC and BD, respectively.

By the definition of frame operator S_Λ and S_Γ we have

$$S_\Lambda x = \sum_{i\in I} \Lambda_i^* \Lambda_i x, \forall x \in \mathcal{H}.$$

$$S_\Gamma y = \sum_{j\in J} \Gamma_j^* \Gamma_j y, \forall y \in \mathcal{K}.$$

Therefore

$$\begin{aligned}
(S_\Lambda \otimes S_\Gamma)(x \otimes y) &= S_\Lambda x \otimes S_\Gamma y \\
&= \sum_{i\in I} \Lambda_i^* \Lambda_i x \otimes \sum_{j\in J} \Gamma_j^* \Gamma_j y \\
&= \sum_{i\in I, j\in J} \Lambda_i^* \Lambda_i x \otimes \Gamma_j^* \Gamma_j y \\
&= \sum_{i\in I, j\in J} (\Lambda_i^* \otimes \Gamma_j^*)(\Lambda_i x \otimes \Gamma_j y) \\
&= \sum_{i\in I, j\in J} (\Lambda_i^* \otimes \Gamma_j^*)(\Lambda_i \otimes \Gamma_j)(x \otimes y) \\
&= \sum_{i\in I, j\in J} (\Lambda_i \otimes \Gamma_j)^*)(\Lambda_i \otimes \Gamma_j)(x \otimes y).
\end{aligned}$$

Now by the uniqueness of frame operator, the last expression is equal to $S_{\Lambda\otimes\Gamma}(x \otimes y)$. Consequently we have $(S_\Lambda \otimes S_\Gamma)(x \otimes y) = S_{\Lambda\otimes\Gamma}(x \otimes y)$. The last equality is satisfied for every finite sum of elements in $\mathcal{H} \otimes_{alg} \mathcal{K}$ and then it's satisfied for all $z \in \mathcal{H} \otimes \mathcal{K}$. It shows that $(S_\Lambda \otimes S_\Gamma)(z) = S_{\Lambda\otimes\Gamma}(z)$. So $S_{\Lambda\otimes\Gamma} = S_\Lambda \otimes S_\Gamma$. □

Theorem 0.15. *Assume that $Q \in End_{\mathcal{A}}^*(\mathcal{H})$ is invertible and $\{\Lambda_i\}_{i\in I} \subset End_{\mathcal{A}\otimes\mathcal{B}}^*(\mathcal{H} \otimes \mathcal{K})$ is a K-operator frame for $\mathcal{H} \otimes \mathcal{K}$ with lower and upper operator frame bounds A and B respectively and frame operator S. If K commute with $Q \otimes \mathcal{I}$, then $\{\Lambda_i(Q^* \otimes \mathcal{I})\}_{i\in I}$ is a*

K-operator frame for $\mathcal{H} \otimes \mathcal{K}$ with lower and upper operator frame bounds $\|Q^{*-1}\|^{-2}A$ and $\|Q\|^2 B$ respectively and frame operator $(Q \otimes \mathcal{I})S(Q^* \otimes \mathcal{I})$.

Proof. Since $Q \in End^*_{\mathcal{A}}(\mathcal{H})$, $Q \otimes \mathcal{I} \in End^*_{\mathcal{A}\otimes\mathcal{B}}(\mathcal{H} \otimes \mathcal{K})$ with inverse $Q^{-1} \otimes \mathcal{I}$. It is obvious that the adjoint of $Q \otimes \mathcal{I}$ is $Q^* \otimes \mathcal{I}$. An easy calculation shows that for every elementary tensor $x \otimes y$,

$$\begin{aligned}\|(Q \otimes \mathcal{I})(x \otimes y)\|^2 &= \|Q(x) \otimes y\|^2 \\ &= \|Q(x)\|^2 \|y\|^2 \\ &\leq \|Q\|^2 \|x\|^2 \|y\|^2 \\ &= \|Q\|^2 \|x \otimes y\|^2.\end{aligned}$$

So $Q \otimes \mathcal{I}$ is bounded, and therefore it can be extended to $\mathcal{H} \otimes \mathcal{K}$. Similarly for $Q^* \otimes \mathcal{I}$, hence $Q \otimes \mathcal{I}$ is $\mathcal{A} \otimes \mathcal{B}$-linear, adjointable with adjoint $Q^* \otimes \mathcal{I}$. Hence for every $z \in \mathcal{H} \otimes \mathcal{K}$ we have by lemma 0.4

$$\|Q^{*-1}\|^{-1}.|z| \leq |(Q^* \otimes \mathcal{I})z| \leq \|Q\|.|z|.$$

By the definition of K-operator frames we have

$$A\langle K^* z, K^* z\rangle_{\mathcal{A}\otimes\mathcal{B}} \leq \sum_{i\in I} \langle \Lambda_i z, \Lambda_i z\rangle_{\mathcal{A}\otimes\mathcal{B}} \leq B\langle z, z\rangle_{\mathcal{A}\otimes\mathcal{B}}.$$

Then

$$\begin{aligned}A\langle K^*(Q^* \otimes \mathcal{I})z, K^*(Q^* \otimes \mathcal{I})z\rangle_{\mathcal{A}\otimes\mathcal{B}} &\leq \sum_{i\in I} \langle \Lambda_i (Q^* \otimes \mathcal{I})z, \Lambda_i (Q^* \otimes \mathcal{I})z\rangle_{\mathcal{A}\otimes\mathcal{B}} \\ &\leq B\langle (Q^* \otimes \mathcal{I})z, (Q^* \otimes \mathcal{I})z\rangle_{\mathcal{A}\otimes\mathcal{B}} \\ &\leq \|Q\|^2 B\langle z, z\rangle_{\mathcal{A}\otimes\mathcal{B}}.\end{aligned}$$

Or

$$\begin{aligned}A\langle K^*(Q^* \otimes \mathcal{I})z, K^*(Q^* \otimes \mathcal{I})z\rangle_{\mathcal{A}\otimes\mathcal{B}} &= A\langle (Q^* \otimes \mathcal{I})K^* z, (Q^* \otimes \mathcal{I})K^* z\rangle_{\mathcal{A}\otimes\mathcal{B}} \\ &\geq \|Q^{*-1}\|^{-2} A\langle K^* z, K^* z\rangle_{\mathcal{A}\otimes\mathcal{B}}.\end{aligned}$$

So we have

$$\|Q^{*-1}\|^{-2} A\langle K^* z, K^* z\rangle_{\mathcal{A}\otimes\mathcal{B}} \leq \sum_{i\in I} \langle \Lambda_i (Q^* \otimes \mathcal{I})z, \Lambda_i (Q^* \otimes \mathcal{I})z\rangle_{\mathcal{A}\otimes\mathcal{B}} \leq \|Q\|^2 B\langle z, z\rangle_{\mathcal{A}\otimes\mathcal{B}}.$$

Now

$$\begin{aligned}(Q\otimes\mathcal{I})S(Q^*\otimes\mathcal{I}) &= (Q\otimes\mathcal{I})(\sum_{i\in I}\Lambda_i^*\Lambda_i)(Q^*\otimes\mathcal{I})\\ &= \sum_{i\in I}(Q\otimes\mathcal{I})\Lambda_i^*\Lambda_i(Q^*\otimes\mathcal{I})\\ &= \sum_{i\in I}(\Lambda_i(Q^*\otimes\mathcal{I}))^*\Lambda_i(Q^*\otimes\mathcal{I}).\end{aligned}$$

Which completes the proof. □

Theorem 0.16. *Assume that $Q\in End_{\mathcal{B}}^*(\mathcal{K})$ is invertible and $\{\Lambda_i\}_{i\in I}\subset End_{\mathcal{A}\otimes\mathcal{B}}^*(\mathcal{H}\otimes\mathcal{K})$ is a K-operator frame for $\mathcal{H}\otimes\mathcal{K}$ with lower and upper operator frame bounds A and B respectively and frame operator S. If K commute with $\mathcal{I}\otimes Q$, then $\{\Lambda_i(\mathcal{I}\otimes Q^*)\}_{i\in I}$ is a K-operator frame for $\mathcal{H}\otimes\mathcal{K}$ with lower and upper operator frame bounds $\|Q^{*-1}\|^{-2}A$ and $\|Q\|^2B$ respectively and frame operator $(\mathcal{I}\otimes Q)S(\mathcal{I}\otimes Q^*)$.*

Proof. Similar to the proof of the previous theorem. □

Studying K-operator frame in Hilbert C^*-modules with different C^*-algebras is interesting and important. In the following theorem we study this situation.

Theorem 0.17. *Let $(\mathcal{H},\mathcal{A},\langle .,.\rangle_{\mathcal{A}})$ and $(\mathcal{H},\mathcal{B},\langle .,.\rangle_{\mathcal{B}})$ be two Hilbert $\mathcal{C}^*$-modules and let $\varphi:\mathcal{A}\longrightarrow\mathcal{B}$ be a $*$-homomorphism and θ be a map on $\mathcal{H}$ such that $\langle\theta x,\theta y\rangle_{\mathcal{B}}=\varphi(\langle x,y\rangle_{\mathcal{A}})$ for all $x,y\in\mathcal{H}$. Also, suppose that $\{\Lambda_i\}_{i\in I}\subset End_{\mathcal{A}}^*(\mathcal{H})$ is a K-operator frame for $(\mathcal{H},\mathcal{A},\langle .,.\rangle_{\mathcal{A}})$ with frame operator $S_{\mathcal{A}}$ and lower and upper operator frame bounds A, B respectively. If θ is surjective, $\theta K^*=K^*\theta$, $\theta\Lambda_i=\Lambda_i\theta$ and $\theta\Lambda_i^*=\Lambda_i^*\theta$ for each i in I, then $\{\Lambda_i\}_{i\in I}$ is a K-operator frame for $(\mathcal{H},\mathcal{B},\langle .,.\rangle_{\mathcal{B}})$ with frame operator $S_{\mathcal{B}}$ and lower and upper operator frame bounds A, B respectively, and $\langle S_{\mathcal{B}}\theta x,\theta y\rangle_{\mathcal{B}}=\varphi(\langle S_{\mathcal{A}}x,y\rangle_{\mathcal{A}})$.*

Proof. Let $y\in\mathcal{H}$ then there exists $x\in\mathcal{H}$ such that $\theta x=y$ (θ is surjective). By the definition of K-operator frames we have

$$A\langle K^*x,K^*x\rangle_{\mathcal{A}}\leq\sum_{i\in I}\langle\Lambda_ix,\Lambda_ix\rangle_{\mathcal{A}}\leq B\langle x,x\rangle_{\mathcal{A}}.$$

By lemma 0.5 we have

$$\varphi(A\langle K^*x,K^*x\rangle_{\mathcal{A}})\leq\varphi(\sum_{i\in I}\langle\Lambda_ix,\Lambda_ix\rangle_{\mathcal{A}})\leq\varphi(B\langle x,x\rangle_{\mathcal{A}}).$$

By the definition of $*$-homomorphism we have

$$A\varphi(\langle K^*x, K^*x\rangle_{\mathcal{A}}) \leq \sum_{i\in I}\varphi(\langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}}) \leq B\varphi(\langle x, x\rangle_{\mathcal{A}}).$$

By the relation betwen θ and φ we get

$$A\langle \theta K^*x, \theta K^*x\rangle_{\mathcal{B}} \leq \sum_{i\in I}\langle \theta\Lambda_i x, \theta\Lambda_i x\rangle_{\mathcal{B}} \leq B\langle \theta x, \theta x\rangle_{\mathcal{B}}.$$

By the relation betwen θ, K^* and Λ_i we have

$$A\langle K^*\theta x, K^*\theta x\rangle_{\mathcal{B}} \leq \sum_{i\in I}\langle \Lambda_i\theta x, \Lambda_i\theta x\rangle_{\mathcal{B}} \leq B\langle \theta x, \theta x\rangle_{\mathcal{B}}.$$

Then

$$A\langle K^*y, K^*y\rangle_{\mathcal{B}} \leq \sum_{i\in I}\langle \Lambda_i y, \Lambda_i y\rangle_{\mathcal{B}} \leq B\langle y, y\rangle_{\mathcal{B}}, \forall y \in \mathcal{H}.$$

On the other hand we have

$$\begin{aligned}
\varphi(\langle S_{\mathcal{A}}x, y\rangle_{\mathcal{A}}) &= \varphi(\langle \sum_{i\in I}\Lambda_i^*\Lambda_i x, y\rangle_{\mathcal{A}}) \\
&= \sum_{i\in I}\varphi(\langle \Lambda_i x, \Lambda_i y\rangle_{\mathcal{A}}) \\
&= \sum_{i\in I}\langle \theta\Lambda_i x, \theta\Lambda_i y\rangle_{\mathcal{B}} \\
&= \sum_{i\in I}\langle \Lambda_i\theta x, \Lambda_i\theta y\rangle_{\mathcal{B}} \\
&= \langle \sum_{i\in I}\Lambda_i^*\Lambda_i\theta x, \theta y\rangle_{\mathcal{B}} \\
&= \langle S_{\mathcal{B}}\theta x, \theta y\rangle_{\mathcal{B}}.
\end{aligned}$$

Which completes the proof. □

Duals of K-operator frame

In the following we define the Dual K-operator frame and we give some properties.

Definition 0.18. Let $K \in End_{\mathcal{A}}^*(\mathcal{H})$ and $\{\Lambda_i \in End_{\mathcal{A}}^*(\mathcal{H}), i \in I\}$ be a K-operator frame for the Hilbert $\mathcal{A}$-module $\mathcal{H}$. An operator Bessel sequences $\{\Gamma_i \in End_{\mathcal{A}}^*(\mathcal{H}), i \in I\}$ is called a K-duals operator frame for $\{\Lambda_i\}_{i\in I}$ if

$$Kf = \sum_{i\in I}\Lambda_i^*\Gamma_i f$$

for all $f \in \mathcal{H}$.

Example 0.19. Let $K\in End_{\mathcal{A}}{}^{*}(\mathcal{H})$ be a surjective operator and $\{\Lambda_i \in End_{\mathcal{A}}^{*}(\mathcal{H}), i \in I\}$ be a K-operator frame for $\mathcal{H}$ with K-frame operator S, then S is invertible.

For all $f \in \mathcal{H}$ we have :

$Sf = \sum_{i\in I} \Lambda_i^{*}\Lambda_i f$.

So $Kf = \sum_{i\in I} \Lambda_i^{*}\Lambda_i S^{-1}Kf$.

Then the sequence $\{\Lambda_i S^{-1}K \in End_{\mathcal{A}}^{*}(\mathcal{H}), i \in I\}$ is a dual K-operator frame of $\{\Lambda_i \in End_{\mathcal{A}}^{*}(\mathcal{H}), i \in I\}$

Theorem 0.20. *Let $K \in \mathrm{End}_{\mathcal{A}}^{*}(\mathcal{H})$ with closed range and let $\{\Lambda_i\}_{i\in I}$ be K-operator frame for $\mathrm{End}_{\mathcal{A}}^{*}(\mathcal{H})$ with frame operator S and frame bounds A and B respectively. Then $\{\Lambda_i \pi_{S(Ran(K))}\left(S^{-1}_{|Ran(K)}\right)^{*}K\}_{i\in I}$ is a K-dual of $\{\Lambda_i\}_{i\in I}$*

Proof. Let $\{\Lambda_i\}_{i\in I}$ be a K-operator frame for $\mathrm{End}_{\mathcal{A}}^{*}(\mathcal{H})$. Since $S : Ran(K) \to S(Ran(K))$ is invertible, we have

$$\begin{aligned}
Kx &= \left(S^{-1}_{|Ran(K)}S_{|Ran(K)}\right)^{*}Kx \\
&= S_{|Ran(K)}\left(S^{-1}_{|Ran(K)}\right)^{*}Kx \\
&= S\pi_{S(Ran(K))}\left(S^{-1}_{|Ran(K)}\right)^{*}Kx \\
&= \sum_{i\in I}\Lambda_i^{*}\Lambda_i\pi_{S(Ran(K))}\left(S^{-1}_{|Ran(K)}\right)^{*}Kx, \text{ for all } x \in \mathcal{H}.
\end{aligned}$$

Also, we have

$$\begin{aligned}
\sum_{i\in I}\langle\Lambda_i\pi_{S(Ran(K))}\left(S^{-1}\right)^{*}Kx, \Lambda_i\pi_{S(Ran(K))}\left(S^{-1}\right)^{*}Kx\rangle_{\mathcal{A}} &= \sum_{i\in I}\langle\Lambda_i^{*}\Lambda_i\pi_{S(Ran(K))}\left(S^{-1}\right)^{*}Kx, \left(S^{-1}\right)^{*}Kx\rangle_{\mathcal{A}} \\
&= \left\langle S\left(S^{-1}\right)^{*}Kx, \left(S^{-1}\right)^{*}Kx\right\rangle \\
&= \left\langle Kx, \left(S^{-1}\right)^{*}Kx\right\rangle \\
&\leq A^{-1}\|K\|^2\left\|K^{\dagger}\right\|^2\langle x, x\rangle_A, x \in \mathcal{H}
\end{aligned}$$

Hence $\left\{\Lambda_i\pi_{Ran(K)}\left(S^{-1}\right)^{*}K\right\}_{i\in I}$ is a dual of the K-operator frame $\{\Lambda_i\}_{i\in I}$. □

Theorem 0.21. *Let $K \in \mathrm{End}_{\mathcal{A}}^{*}(\mathcal{H})$ with closed range, and $\{\Lambda_i\}_{i\in I}$ be K-operator frame for $\mathrm{End}_{\mathcal{A}}^{*}(\mathcal{H})$ with frame bounds A and B. Then, there is one to one corespondence between K-dual of $\{\Lambda_i\}_{i\in I}$ and operator $\varphi \in \mathrm{End}_{\mathcal{A}}^{*}(\mathcal{H}, \ell^2(\mathcal{H}))$ such that $R^{*}\varphi = 0$*

Proof. Let $\{\Gamma_i\}_{i\in I}$ be a K-dual of K-operator frame $\{\Lambda_i\}_{j\in I}$ with frame bounds C and D and S be its frame operator. Define a mapping $\varphi : \mathcal{H} \to \ell^2(\mathcal{H})$ by

$$(\varphi x)_i = \Gamma_i x - \Lambda_i\pi_{S(Ran(K))}\left(S^{-1}\right)^{*}Kx, x \in \mathcal{H}$$

Then φ is adjointable operator on $\mathcal{H}$. and we have for each $x \in \mathcal{H}$ we have

$$\begin{aligned}
\| \{(\varphi x)_i\}_{i\in I} \| &= \|\{\Gamma_i x - \Lambda_i \pi_{S(Ran(K))} \left(S^{-1}\right)^* Kx\}_{i\in I}\| \\
&\leq \|\{\Gamma_i x\}_{i\in I}\| + \|\{\Lambda_i \pi_{S(Ran(K))} \left(S^{-1}\right)^* Kx\}_{i\in I}\| \\
&= \|\sum_{i\in I} \langle \Gamma_i x, \Gamma_i x\rangle_{\mathcal{A}}\|^{1/2} + \|\sum_{i\in I} \langle \Lambda_i \pi_{S(Ran(K))} \left(S^{-1}\right)^* Kx, \Lambda_i \pi_{S(Ran(K))} \left(S^{-1}\right)^* Kx\rangle_{\mathcal{A}}\|^{1/2} \\
&= \|\sum_{i\in I} \langle \Gamma_i x, \Gamma_i x\rangle_{\mathcal{A}}\|^{1/2} + \|\sum_{i\in I} \langle S \left(S^{-1}\right)^* Kx, \left(S^{-1}\right)^* Kx\rangle_{\mathcal{A}}\|^{1/2} \\
&\leq C^{1/2} \langle x, x\rangle_{\mathcal{A}}^{1/2} + A^{-1/2} \|K\| \left\|K^\dagger\right\| \langle x, x\rangle_{\mathcal{A}}^{1/2}
\end{aligned}$$

Also, we have

$$\begin{aligned}
R^* \varphi x &= \sum_{i\in I} \Lambda_i^* (\varphi x)_i \\
&= \sum_{i\in I} \Lambda_i^* \left(\Gamma_i^* x - \Lambda_i \pi_{S(Ran(K))} \left(S^{-1}\right)^* Kx\right) \\
&= Kx - Kx \\
&= 0, \text{ for all } x \in \mathcal{H}
\end{aligned}$$

Conversely, let $\varphi \in B\left(\mathcal{H}, \ell^2(\mathcal{H})\right)$ and $R^* \varphi = 0$.

$$\Gamma_i x = \Lambda_i \pi_{S(Ran(K))} \left(S^{-1}\right)^* Kx + (\varphi x)_i, x \in \mathcal{H}$$

Then, we have

$$\begin{aligned}
\|\{\Gamma_i x\}_{i\in I}\| &\leq \left\|\{\Lambda_i \pi_{S(Ran(K))} \left(S^{-1}\right)^* Kx\}_i\right\| + \|\{(\varphi x)_i\}_i\| \\
&\leq A^{-1} \|K\|^2 \left\|K^\dagger\right\|^2 \langle x, x\rangle_{\mathcal{A}} + \|\varphi\|^2 \langle x, x\rangle_{\mathcal{A}} \\
&\leq \left(A^{-1} \|K\|^2 \left\|K^\dagger\right\|^2 + \|\varphi\|^2\right) \langle x, x\rangle_{\mathcal{A}}
\end{aligned}$$

Therefore, $\{\Gamma_i\}_{i\in I}$ is an operator Bessel sequence. Also, we have

$$\begin{aligned}
\sum_{i\in I} \Lambda_i^* \Gamma_i x &= \sum_{i\in I} \Lambda_i^* \Lambda_i \pi_{S(Ran(K))} \left(S^{-1}\right)^* Kx + \sum_{i\in I} \Lambda_i^* (\varphi x)_i \\
&= Kx + R^* \varphi x = Kx, x \in \mathcal{H}
\end{aligned}$$

Hence, $\{\Gamma_i\}_{i\in I}$ is a K-dual of the K-operator frame $\{\Lambda_i\}_{i\in I}$. □

Theorem 0.22. *Let $\{\Lambda_i\}_{i\in I}$ and $\{\Gamma_j\}_{j\in J}$ are K-operator frame and L-operator frame respectively in $\mathcal{H}$ and $\mathcal{K}$, with duals $\{\tilde{\Lambda}_i\}_{i\in I}$ and $\{\tilde{\Gamma}_j\}_{j\in J}$ respectively, then $\{\tilde{\Lambda}_i \otimes \tilde{\Gamma}_j\}_{i,j\in I,J}$ is a dual of $\{\Lambda_i \otimes \Gamma_j\}_{i,j\in I,J}$.*

Proof. By definition, $\forall x \in \mathcal{H}$ and $\forall y \in \mathcal{K}$ we have:

$$\sum_{i\in I}\Lambda_i^*\tilde{\Lambda}_i x = Kx$$

$$\sum_{j\in J}\Gamma_j^*\tilde{\Gamma_j} y = Ly$$

then :

$$(K\otimes L)(x\otimes y) = Kx\otimes Ly = \sum_{i\in I}\Lambda_i^*\tilde{\Lambda}_i x\otimes\sum_{j\in J}\Gamma_j^*\tilde{\Gamma_j} y$$

$$\sum_{i\in I}\Lambda_i^*\tilde{\Lambda}_i x\otimes\sum_{j\in J}\Gamma_j^*\tilde{\Gamma_j} y = \sum_{i,j\in I,J}\Lambda_i^*\tilde{\Lambda}_i x\otimes\Gamma_j^*\tilde{\Gamma_j} y$$

$$\sum_{i\in I}\Lambda_i^*\tilde{\Lambda}_i x\otimes\sum_{j\in J}\Gamma_j^*\tilde{\Gamma_j} y = \sum_{i,j\in I,J}(\Lambda_i^*\otimes\Gamma_j^*).(\tilde{\Lambda}_i x\otimes\tilde{\Gamma_j} y)$$

$$\sum_{i\in I}\Lambda_i^*\tilde{\Lambda}_i x\otimes\sum_{j\in J}\Gamma_j^*\tilde{\Gamma_j} y = \sum_{i,j\in I,J}(\Lambda_i\otimes\Gamma_j)^*.(\tilde{\Lambda}_i\otimes\tilde{\Gamma_j}).(x\otimes y)$$

then $\{\tilde{\Lambda}_i\otimes\tilde{\Gamma_j}\}_{i,j\in I,J}$ is a dual of $\{\Lambda_i\otimes\Gamma_j\}_{i,j\in I,J}$ □

Corollary 0.23. *Let $\{\Lambda_{i,j}\}_{0\le i\le n;j\in J}$ be a family of K_i-operator frames, such $0\le i\le n$ and $\{\tilde{\Lambda}_{i,j}\}_{0\le i\le n;j\in J}$ their dual, then $\{\tilde{\Lambda}_{0,j}\otimes\tilde{\Lambda}_{1,j}\otimes ......\otimes\tilde{\Lambda}_{n,j}\}_{j\in J}$ is a dual of $\{\Lambda_{0,j}\otimes\Lambda_{1,j}\otimes ......\otimes\Lambda_{n,j}\}_{j\in J}$.*

DECLARATIONS

Availablity of data and materials

Not applicable.

Competing interest

The authors declare that they have no competing interests.

Fundings

Authors declare that there is no funding available for this article.

Authors' contributions

The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

Acknowledgements

The authors are thankful to the area editor and referees for giving valuable comments and suggestions

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