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The Primitive-Solutions of Diophantine Equation $x^2 + pqy^2 = z^2$, for primes p, q

Solusi Primitif Persamaan Diophantine $x^2 + pqy^2 = z^2$ untuk bilangan-bilangan prima p dan q

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Abstract

In this paper, we determine the primitive-solutions of diophantine equations $x^2 + pqy^2 = z^2$, for positive integers x, y, z and primes p, q. Our work is based on the development of the previous results, namely using the solutions of the Diophantine equation $x^2 + y^2 = z^2$, and looking characteristics of the solutions of the Diophantine equation $x^2 + 3y^2 = z^2$ and $x^2 + 9y^2 = z^2$.

Keywords: composite number, diophantine equation, prime number, primitive solution.

1. INTRODUCTION

A Diophantine equation is an equation of the form

$$f(x_1, x_2, \dots, x_n) = 0, \qquad 1.1$$

where f is an n-variable function with $n \geq 2$. The solution of Equation (1.1) is an n-uple $x_1, x_2, \dots x_n$ satisfying the equation [3]. For example, 14,223 is one solution of Diophantine equation $17x + 8y = 2021$, and 3,4,5 is the solution of Diophantine equation $x^2 + y^2 = z^2$.

Nowadays, there have been many studies about Diophantine equations. Most of their research is about finding the solutions of a given equation, one of which is the work on the equation $x^2 + 3^a 41^b = y^n$ by Alan and Zengin [2] where a, b are non-negative integers and x, y are realtively prime. There are many forms of Diophantine equations with various variables defined. Rahmawati et al [7] figured out the solutions from the equation $(7^k - 1)^x + (7^k)^y = z^2$ where x, y, and z are non-negative integers and k is the positive even integer, Burshtein [4] stated the solutions of Diophantine equation $p^x + p^y = z^4$ when $p \geq 2$ are primes and x, y, z are positive integers, and Chakraborty and Hoque [5] investigated the solvability of the Diophantine equation $dx^2 + p^{2a}q^{2b} = 4y^p$, where $d > 1$ is a square-free integer, p, q are distinct odd primes and x, y, a, b are positive integers with $gcd(x, y) = 1$.

Another interesting Diophantine equation is $x^2 + cy^2 = z^2$, where all the variables are integers. Some cases of this problem have been solved, such as for case of $c = 1$ (see in [8]).

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Next, there are Abdealim and Dyani [1] who had given the solutions for case of $c = 3$ by using the arithmetic technical. Following this, Rahman and Hidayat [6] presented the primitive-solutions for case of $c = 9$ using characteristics of the primitive solutions which are a development of the previous cases.

On this paper, we extend the results of [1], [6] and [8] to determine the primitive-solutions of Diophantine equation $x^2 + pqy^2 = z^2$ where x, y and z are positive integers, and p and q are primes. We establish results that the equation for case y is odd has no primitive-solution and case y is even have two primitive-solutions.

2. MAIN RESULTS

Before showing our results, firstly, we fix some notation. If not previously defined, then we use Diophantine equation $x^2 + pqy^2 = z^2$ with x, y, z are positive integers, and p, q are primes. Also, if integers m and n are relatively primes, we write $(m, n) = 1$. Sometimes, we just write x, y for indicate x and y.

Definition 2.1. *Any triple Phytagoras x, y, z is called a triple primitive Phytagoras if$(x, y, z) = 1$* [3].

Next, We note one result from [3],

Theorem 2.2. *The positive integers x, y, z is a primitive-solution of Diophantine equation $x^2 + y^2 = z^2$ with y is even, if and only if there are postive integers m and n such that $x = m^2 - n^2$, $y = 2mn$, and $z = m^2 + n^2$ with $(m, n) = 1$, $m > n$, and m, n have different parity.*

We also share the fundamental theorem of arithmetic without any comment,

Theorem 2.3. *Every positive integer can be written uniquely as a product of primes, with the prime factors in the product written in order of non-decreasing size* [3].

Now, we begin our work.

Definition 2.4. The positive integers x, y, z is called a primitive-solution of Diophantine equation $x^2 + pqy^2 = z^2$ if $(x, y, z) = 1$.

Example 2.5. 2,1,45 is a primitive-solution of Diophantine equation $x^2 + 2021y^2 = z^2$, because of $2^2 + 2021(1)^2 = 2025 = 45^2$ and $(2,1,45) = 1$.

Theorem 2.6. *If x, y, z is a solution of Diophantine equation $x^2 + pqy^2 = z^2$ with $(x, y, z) = d$ such that $x = dx_1, y = dy_1$, and $z = dz_1$ for integers x_1, y_1, z_1, then x_1, y_1, z_1 is a solution of Diophantine equation $x^2 + pqy^2 = z^2$ with $(x_1, y_1, z_1) = 1$.*

Proof. Let integers x, y, z is a solution of Diophantine equation $x^2 + pqy^2 = z^2$, so

$$x^2 + pqy^2 = z^2$$
$$(dx_1)^2 + pq(dy_1)^2 = (dz_1)^2$$
$$d^2(x_1^2 + pqy_1^2) = d^2z_1^2$$
$$x_1^2 + pqy_1^2 = z_1^2 \quad 2.1$$

From Equation (2.1), we can conclude that x_1, y_1, z_1 is a solution of Diophantine equation $x^2 + pqy^2 = z^2$. Also, from $(x, y, z) = d$, we have $\left(\frac{x}{d}, \frac{y}{d}, \frac{z}{d}\right) = 1$. This is equal to $(x_1, y_1, z_1) = 1$ which completes the proof of Theorem 2.3.

Example 2.7. 4,2,90 is a solution of Diophantine equation $x^2 + 2021y^2 = z^2$. We have $(4,2,90) = 2$. Hence, we get $x_1 = 2$, $y_1 = 1$ and $z_1 = 45$. From Example 2.5, we have 2,1,45 is also the solution of the equation with $(2,1,45) = 1$.

Lemma 2.8. *If the integers x, y, z is a primitive-solution of Diophantine equation $x^2 + pqy^2 = z^2$, then (x,y)=(y,z)=(x,z)=1.*

Proof. Suppose $(x, y) \neq 1$, then there a prime p_1 with $p_1 = (x, y)$ so that $p|x$ and $p|y$. Therefore, $p_1|(x^2 + pqy^2 = z^2)$. Hence, $p_1|z^2$ and then $p_1|z$. Because $p_1|x$, $p_1|y$ and $p_1|z$, we can conclude that $(x, y, z) = p_1$. This contradicts the fact that x, y, z is a primitive-solution of Diophantine equation $x^2 + pqy^2 = z^2$. Consequently, it must be $(x, y) = 1$. Using similar techniques, we prove for $(y, z) = 1$ and $(x, z) = 1$.

Theorem 2.9. *If the positive integers $x. y, z$ is a primitve-solution of Diophantine equation $x^2 + pqy^2 = z^2$and y is even, then x dan z are odd.*

Proof. Let y is even and x, y, z is a primitive-solution of Diophantine equation $x^2 + pqy^2 = z^2$. Using Lemma 2.8, we have $(x, y) = 1$ and $(y, z) = 1$. These equations mean that x and z are odd.

Example 2.10. 95,92,4137 is the primitive-solution of Diophantine equation $x^2 + 2021y^2 = z^2$ where $y = 92$ is even, and $x = 95$ and $z = 4137$ are odd.

Theorem 2.11. *If the positive integers $x. y, z$ is a primitve-solution of Diophantine equation $x^2 + pqy^2 = z^2$and y is odd, then x dan z are even.*

Proof. Let y is odd and x, y, z is a primitive-solution of Diophantine equation $x^2 + pqy^2 = z^2$. Using Lemma 2.8, we have $(x, y) = 1$ and $(y, z) = 1$. These equations mean that x and z are even.

Theorem 2.12. If r, s, t are positive integers with $(r, s) = 1$ and $rs = pqt^2$ where p, q are primes, then there are integers m and n such that

i. *r= $pqm^2$and $s = n^2$,*
ii. *$r = m^2$ and $s = pqn^2$, or*
iii. *$r = pm^2$ and $s = qn^2$.*

Proof. Based on Theorem 2.3, we can write each positive integers r, s, and t as a single product of their primes. Write $r = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_u^{\alpha_u}$, $s = p_{u+1}^{\alpha_{u+1}} p_{u+2}^{\alpha_{u+2}} \dots p_v^{\alpha_v}$, and $t = q_1^{\beta_1} p_2^{\beta_2} \dots p_k^{\beta_k}$. So, we get $pqt^2 = pqq_1^{2\beta_1} q_2^{2\beta_2} \dots q_k^{2\beta_k}$. Since $(r, s) = 1$, It means that prime factors of r and s are different. Because $rs = pqt^2$, we get

$$\left(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_u^{\alpha_u}\right)\left(p_{u+1}^{\alpha_{u+1}} p_{u+2}^{\alpha_{u+2}} \dots p_v^{\alpha_v}\right) = pqq_1^{2\beta_1} q_2^{2\beta_2} \dots q_k^{2\beta_k}. \quad 2.2$$

Case 1. $p = q$

If $p = q$, we can write $pq = q_{k+1}^{2\beta_{k+1}}$ where $\beta_{k+1} = 1$. Hence, we can write Equation (2.2) as the following

$$\left(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_u^{\alpha_u}\right)\left(p_{u+1}^{\alpha_{u+1}} p_{u+2}^{\alpha_{u+2}} \dots p_v^{\alpha_v}\right) = q_1^{2\beta_1} q_2^{2\beta_2} \dots q_k^{2\beta_k} q_{k+1}^{2\beta_{k+1}} \quad 2.3$$

If we look at the details on Equations (2.3), two sides of the equation must be equal. Therefore, every p_i has to be equal with q_j, so that $\alpha_i = 2\beta_j$. Hence, every exponent α_i is even. Consequently, $\beta_j = \frac{\alpha_i}{2}$ is an integer.

Let m and n are integers with $m = p_1^{\frac{\alpha_1}{2}} p_2^{\frac{\alpha_2}{2}} \dots p_u^{\frac{\alpha_u}{2}}$ and $n = p_{u+1}^{\frac{\alpha_{u+1}}{2}} p_{u+2}^{\frac{\alpha_{u+2}}{2}} \dots p_v^{\frac{\alpha_v}{2}}$. So,

$$pqq_1^{2\beta_1}q_2^{2\beta_2}\dots q_k^{2\beta_k} = pq\left(p_1^{\frac{\alpha_1}{2}} p_2^{\frac{\alpha_2}{2}} \dots p_u^{\frac{\alpha_u}{2}}\right)^2 \left(p_{u+1}^{\frac{\alpha_{u+1}}{2}} p_{u+2}^{\frac{\alpha_{u+2}}{2}} \dots p_v^{\frac{\alpha_v}{2}}\right)^2$$
$$pqt^2 = pqm^2n^2$$
$$pqt^2 = pqm^2n^2$$
$$pqt^2 = (pqm^2)(n^2)$$
$$pqt^2 = (m^2)(pqn^2)$$
$$pqt^2 = (pm^2)(qn^2)$$

Case 2. $p \neq q$

If $p \neq q$, then there are two p_i which are equal to each p and q. Suppose both are $p_c = p$ and $p_c = q$. Then, Equation (2.2) can be written as the following

$$p_1^{\alpha_1}p_2^{\alpha_2}\dots p_c^{\alpha_c}p_d^{\alpha_d}\dots p_u^{\alpha_u}p_{u+1}^{\alpha_{u+1}}\dots p_v^{\alpha_v} = pqq_1^{2\beta_1}q_2^{2\beta_2}\dots q_k^{2\beta_k}$$
$$p_1^{\alpha_1}p_2^{\alpha_2}\dots p_c^{\alpha_f}p_d^{\alpha_g}\dots p_u^{\alpha_u}p_{u+1}^{\alpha_{u+1}}\dots p_v^{\alpha_v} = q_1^{2\beta_1}q_2^{2\beta_2}\dots q_k^{2\beta_k} \qquad 2.4$$

where $\alpha_f = \alpha_c - 1$ and $\alpha_g = \alpha_d - 1$. For a note, positions of p_c and p_d in Equation (2.4) can be randomly in r or s. We don't go into detail about them because they will give the same result later. Using similar techniques in Case 1, we get every exponent α_i in Equation (2.4) is even. Hence, $\beta_j = \frac{\alpha_i}{2}$ is an integer.

Let m and n are integers with $m = p_1^{\frac{\alpha_1}{2}} p_2^{\frac{\alpha_2}{2}} \dots p_c^{\frac{\alpha_f}{2}} p_d^{\frac{\alpha_g}{2}} \dots p_u^{\frac{\alpha_u}{2}}$ and $n = p_{u+1}^{\frac{\alpha_{u+1}}{2}} p_{u+2}^{\frac{\alpha_{u+2}}{2}} \dots p_v^{\frac{\alpha_v}{2}}$. So,

$$pqq_1^{2\beta_1}q_2^{2\beta_2}\dots q_k^{2\beta_k} = pq\left(p_1^{\frac{\alpha_1}{2}} p_2^{\frac{\alpha_2}{2}} \dots p_c^{\frac{\alpha_f}{2}} p_d^{\frac{\alpha_g}{2}} \dots p_u^{\frac{\alpha_u}{2}}\right)^2 \left(p_{u+1}^{\frac{\alpha_{u+1}}{2}} p_{u+2}^{\frac{\alpha_{u+2}}{2}} \dots p_v^{\frac{\alpha_v}{2}}\right)^2$$
$$pqt^2 = pqm^2n^2$$
$$pqt^2 = pqm^2n^2$$
$$pqt^2 = (pqm^2)(n^2)$$
$$pqt^2 = (m^2)(pqn^2)$$
$$pqt^2 = (pm^2)(qn^2)$$

Combining Case 1 and Case 2, it has proven that $r = pqm^2$ and $s = n^2$, $r = m^2$ and $s = pqn^2$, or $r = pm^2$ and $s = qn^2$, where r and s are integers.

Example 2.13. Take $p = 3$, $q = 2$ and $t = 5$. Hence, we get $rs = pqt^2 = 150$. Next, we can choose integers m and n to define r and s, such as

i. $m = 5$ and $n = 1$ so that $r = pqm^2 = 150$ and $s = n^2 = 1$,
ii. $m = 1$ and $n = 5$ so that $r = m^2 = 1$ and $s = pqn^2 = 150$, or
iii. $m = 25$ and $n = 1$ so that $r = pm^2 = 75$ and $s = qn^2 = 2$.

It is clear that $rs = 150$ when $r = 150$ and $s = 2$, $r = 1$ and $s = 150$, or $r = 75$ and $s = 2$.

Theorem 2.14. *The Diophantine equation $x^2 + p^2y^2 = z^2$ with y is odd, and p,q are primes have no primitive-solution.*

Proof. Using Theorem 2.11, If y is odd then x and z are even. Hence, $z-x$ and $z+x$ are even. Write $z-x=t_1$ and $z+x=t_2$, for integers t_1,t_2. From the Diophantine equation $x^2+pqy^2=z^2$, we get $pqy^2=(z-x)(z+x)=4t_1t_2$. Because y is odd, pq must divide by 4. The only possible values are $p=2$ and $q=2$. So, we have $4y^2=(z-x)(z+x)$. If $z-x=4y^2$ and $z+x=1$, then $z=\frac{4y^2-1}{2}$ is not an integer. So, this is impossible. Next, if $z-x=1$ and $z+x=4y^2$, then $z=\frac{1-4y^2}{2}$ is not integer. So, this is also impossible. Then, If $z-x=2y$ and $z+x=2y$ then we get $x=0$ but this is also not possible since $(x,y,z)=1$. So, we can conclude that the Diophantine equation $x^2+pqy^2=z^2$ with y is odd don't have primitve-solutions.

After we have proved Theorem 2.14, we will share our results on the case y is even.

In the following theorem, we determine the primitive-solutions of Diophantine equation $x^2+pqy^2=z^2$ for case of $p=q$.

Theorem 2.15. *The positive integers x,y,z is a primitve-solution of Diophantine equation $x^2+p^2y^2=z^2$ with y is even, and p is prime, if and only if $x=m^2-n^2$, $y=\frac{2}{p}mn$, and $z=m^2+n^2$, where $(m,n)=1$, m and n have different parity, $m>n$, and $m=pa$ or $n=pb$ for any integers a,b.*

Proof. ($\Longrightarrow$)Let $t=py$. Because y is even, t is also even. Based on Theorem 2.2, the primitive-solution of Diophantine equation $x^2+t^2=z^2$ such as $x=m^2-n^2$,$t=2mn$, and $z=m^2+n^2$, with $(m,n)=1$, $m>n$, and m,n has different parity. Because $t=py$ and $t=2mn$, we get $y=\frac{2}{p}mn$. Since y is a positive integer and p is prime, mn must be divisible by p. Consequently, $m=pa$ or $n=pb$ for any integers a,b.

($\Longleftarrow$)We will show that x,y,z satisfies the Diophantine equation $x^2+p^2y^2=z^2$.

Case 1. $m=pa$

$$\begin{aligned} x^2+p^2y^2 &= (m^2-n^2)+p^2\left(\frac{2}{p}mn\right)^2 \\ &= (p^2a^2-n)^2+(2pan)^2 \\ &= (p^2a^2+n^2)^2 \\ &= (m^2+n^2)^2 \\ &= z^2. \end{aligned}$$

Case 2. $n=pb$

$$\begin{aligned} x^2+p^2y^2 &= (m^2-n^2)+p^2\left(\frac{2}{p}mn\right)^2 \\ &= (m-p^2b^2)^2+(2mpb)^2 \\ &= (m^2+p^2b^2)^2 \\ &= (m^2+n^2)^2 \\ &= z^2. \end{aligned}$$

So, x,y,z is the solution of Diophantine equation $x^2+p^2y^2=z^2$. Next, integers x,y,z is called primitive if $(x,y,z)=1$. Suppose $(x,y,z)\neq 1$. This means that there is a prime p such that $p=(x,y,z)$. Hence, $p|x$ and $p|z$. Furthermore, $p|(x+z)=2m^2$ and $p|(x-z)=n^2$. Because m and n have different parity, we get $p\neq 2$ so that $p|m^2$ and $p|m$. Also, it is clear that $p|n^2$ and $p|n$. Because $p|m$ and $p|n$, we can conclude that $p=(m,n)$. It contradicts to $(m,n)=1$. However, it must be $(x,y,z)=1$. So, x,y,z is a primitive-solution of Diophantine equation $x^2+p^2y^2=z^2$.

Example 2.16. Take $m = 3$ and $n = 2$. Hence, we get $x = m^2 - n^2 = 5$, $y = \frac{2}{p} mn = 4$ for $p = 3$, and $z = m^2 + n^2 = 13$. It is clear that 5,4,13 is a primitive-solution of Diophantine equation $x^2 + 9y^2 = z^2$.

Theorem 2.17. *The positive integers x, y, z with y is even is a primitive-solution of Diophantine equation $x^2 + pqy^2 = z^2$ if and only if*

i. $x = pqm^2 - n^2$, $y = 2mn$, *and* $z = pqm^2 + n^2$,
ii. $x = m^2 - pqn^2$, $y = 2mn$, *and* $z = m^2 + pqn^2$, *or*
iii. $x = pm^2 - pn^2$, $y = 2mn$, *and* $z = pm^2 + qn^2$,

where $(m, n) = 1$, $m > n$, *and* m, n *has different parity.*

Proof. ($\Longrightarrow$) Based on Theorem 2.9, If y is even, then x and z are odd. Hence, $z + x$ dan $z - x$ are even so that there are two integers $r = \frac{z+x}{2}$ and $s = \frac{z-x}{2}$. Write $y = 2t$, for any integer t. So, we get $x^2 + pq(2t)^2 = z^2$ or $pqt^2 = rs$. Furthermore, using Theorem 2.12, we have

i. $r = pqm^2$ and $s = n^2$,
ii. $r = m^2$ and $s = pqn^2$, or
iii. $r = pm^2$ and $s = qn^2$.

Substituting values of r and s above to the equations $r = \frac{z+x}{2}$, $s = \frac{z-x}{2}$ and $y = 2t$. We get respectively

i. $x = pqm^2 - n^2$, $y = 2mn$, and $z = pqm^2 + n^2$,
ii. $x = m^2 - pqn^2$, $y = 2mn$, and $z = m^2 + pqn^2$, and
iii. $x = pm^2 - pn^2$, $y = 2mn$, and $z = pm^2 + q$.

($\Longleftarrow$)We substitute values of x, y and z to the Diophantine equation $x^2 + pqy^2 = z^2$.

i. $x^2 + pqy^2 = (pqm^2 - n^2)^2 + pq(2mn)^2$
$= p^2q^2m^4 + 2pqm^2n^2 + n^4$
$= (pqm^2 + n^2)^2$
$= z^2$.

ii. $x^2 + pqy^2 = (m^2 - pqn^2)^2 + pq(2mn)^2$
$= m^4 + 2pqm^2n^2 + p^2q^2n^4$
$= (m^2 + pqn^2)^2$
$= z^2$.

iii. $x^2 + pqy^2 = (pm^2 - qn^2)^2 + pq(2mn)^2$
$= p^2m^4 + 2pqm^2n^2 + q^2n^4$
$= (pm^2 + pn^2)^2$
$= z^2$.

Because $(m, n) = 1$, $m > n$, and m, n has different parity, we can conclude that integers x, y, z is a primitive-solution of Diophantine equation $x^2 + pqy^2 = z^2$. Also, from $y = 2mn$, we get y which is even.

Example 2.18. Take $p = 47$, $q = 43$, $m = 2$ and $n = 1$. It is clear that

i. $x = pqm^2 - n^2 = 8083$, $y = 2mn = 4$ and $z = pqm^2 + n^2 = 8085$, and
ii. $x = pm^2 - pn^2 = 145$, $y = 2mn = 4$ and $z = pm^2 + qn^2 = 231$

are two primitive-solutions of Diophantine equation $x^2 + 2021y^2 = z^2$.

Example 2.19. Take $p = 47$, $q = 43$, $m = 46$ and $n = 1$. Hence, we get $x = m^2 - pqn^2 = 95$, $y = 2mn = 92$, and $z = m^2 + pqn^2 = 4137$. From Example 2.10, we get 95,92,4137 is the primitive-solution of Diophantine equation $x^2 + 2021y^2 = z^2$.

REFERENCES

[1] Abdelalim and Dyani, 2014. The Solution of Diophantine Equation $x^2 + 3y^2 = z^2$. International Journal of Algebra, Vol. 8, No. 15, pp. 729-723.

[2] Alan M. and Zengin U. 2020. On the Diophantine equation $x^2 + 3^a 41^b = y^n$. Periodica Mathematica Hungarica, Vol. 81, No. 2, pp. 284–291.

[3] Andreescu T., Andrica D. & Cucurezeanu I., 2010. *An Introduction to Diophantine Equations: A Problem-Based Approach*, Birkhäuser (Springer Science+Business Media LLC), Boston.

[4] Burshtein, N. 2020. Solutions of the Diophantine Equation $p^x + p^y = z^4$ when $p \geq 2$ are is Primes and x, y, z are Positive Integers. Annals of Pure and Applied Mathematics, Vol. 21, No. 2, pp. 125-128.

[5] Chakraborty K. and Hoque A. 2021. On the Diophantine Equation $dx^2 + p^{2a}q^{2b} = 4y^p$. Results in Mathematics, Vol. 77, pp. 18.

[6] Rahman, S.I. & Hidayat, N., 2018. Solusi Primitif Persamaan Diophantine $x^2 + 9y^2 = z^2$. *Prosiding Konferensi Nasional Matematika (KNM)*, XIX, pp. 71-76, Himpunan Matematika Indonesia (IndoMS) Perwakilan Surabaya, Surabaya.

[7] Rahmawati R., Sugandha A., Tripena A. and Prabowo A. 2018. The Solution for the Non linear Diophantine Equation $(7k - 1)^x + (7k)^y = z^2$ with k as the positive even whole number. *Journal of Physics: Conference Series*, Vol. 1179, The 1st International Conference on Computer, Science, Engineering and Technology 27–28 November 2018, Tasikmalaya, Indonesia.

[8] Rosen, K.H. 1984. *Elementary Number Theory and Its Applications*, Perason, Boston.