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Global Attractors for Abstract Evolutionary Equations

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Abstrak

We establish in this paper that under usual assumption on the function in abstract Cauchy problem, the existence of global attractor for the semigroup generated in this problem is merely a consequence of such a priori estimate. It is shown that asymptotic behavior of solutions can controlled by this estimate.

Keywords: *Global attractor, priori estimate, asymptotically independent.*

1. Introduction

This paper is concerned with the study of the existence of a global attractor. The global attractor is a basic concept and tool to study asymptotic behaviors of solutions on evolution equations. It is an invariant compact set absorbing all the bounded sets as time goes to infinity.

A number of parabolic initial boundary value problems originating in the applied sciences can be shown into the class of abstract evolutionary equations:

$$\begin{aligned} \dot{u} + Au &= F(u), \qquad t > 1 \\ u(0) &= u_0 \end{aligned} \tag{1}$$

where A is a sectorial operator with compact resolvent in Banach space X and F is a nonlinear function which Lipschitz continuous on bounded subset of $X^{\propto}$ with $\propto \in [0,1]$.

A particular advantage of studying problems of this type is, that the estimates necessary to ensure the existence of the global solutions. By this estimates, the control over their asymptotic behaviors become much easier to obtain. If the initial data is known, a strongly continuous semigroup corresponding to equation (1) may be defined. From resuts of (Cholewa & Dlotko, 1996), the semigroup $\{T(t)\}$ which generated, maps bounded sets into bounded sets and $\{T(t)\}$ needs to be compact for $t > 0$ and the semigroup is known to be compact if only the main part of the operator has a compact resolvent. Furthermore, the knowledge of any estimate of solutions asymtotically independent of initial conditions and the property that the operator attract every bounded set is then sufficient for the existence of the global attractor. Monograph (Cholewa & Dlotko, 2000) ensure that generally i.e. for problem which admit the formulation (1), point dissipativeness of $\{T(t)\}$ is the crusial step for the construction of global attractor.

Recent years several authors have developed this problem to find the global atrractor for some parabolic equations. They employee some more complicated technique and estimates to show the existence the global atrractor and it"s regularity. Specifically, we cite here paper of (Nakao & Aris, 2007).

Theoritical considerations of the following section are mostly based on papers (Cholewa & Dlotko, 1996; Aris, 2004).

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2. Preliminaries

Let V be a metric space. Without lack of generality, let assume that a set $A \subset V$ is a sectorial positively invariant operator so that the power spaces X^{α} are defined with $\propto \epsilon[0,1)$.

Assumption 1.
$A: D \to (A)$, X be a sectorial and positive operator in a Banach space X i.e. $Re\,\sigma(A) > 0$ and function $F : X \to X^{\propto}$ be Lipschitz continuous on bounded subsets of X^{α} for some $\alpha \in [0,1)$.

Definition 1.
Let X be Banach space, $\propto \epsilon[0,1)$ and u_0 be an element of X^{α}. If, for some real $\tau > 0$, a function $u \in C([0,\tau), X^{\propto})$ satisfies

$u(0)=u_0$,
$u \,\epsilon\, C^1((0,\tau), X))$,
$u(t) \in D(A)$ for each $t \in (0,\tau)$,
the first equation in (1) holds in X for all $t \in (0,\tau)$

then u is called a local X^{α} solution of (1).

Lemma 1.
Let Assumption 1 *hold and $u \in C([0,\tau), X^{\propto})$. Then u is a local X^{α} solution of* (1) *if and only if u satisfies in X the integral equation*:

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)} F(u(s))ds \qquad \text{for } t \in [0,\tau). \tag{2}$$

When we study a nonlinear evolutionary equation, it is very important to understand the behavior of its solution when $t \to \infty$.

Definition 2.
A function u = u(t) is called a global X^{α} solution of (1) *if it fulfills the requirements of Defenition* 1 *with $\tau = +\infty$.*

If any initial data $u_0 \,\epsilon\, X^{\alpha}$ corresponds a global X^{α} solution $u(t, u_0)$ of (1) then a strongly continuous semigroup $\{T(t)\}$ corresponding to (1) may be defined on X^{α} as relation:

$$T(t)\, u_0 = u(t, u_0), \; t > 0 \tag{3}$$

Now, to show the existence of the global attractor, we need to consider two conditions of the semigroup:

(C1). *Relation* (3) *defines on X^{α}, corresponding to* (1) *a strongly continuous semigroup* $\{T(t)\}$ *of global X^{α} solution having orbits of bounded sets bounded.*

(C2). *It is possible to choose*

- *a Banach space Y with $D(A) \subset Y$,*
- *a non decreasing function $g : [0, +\infty) \to [0, +\infty)$,*
- *a locally function $c : [0, +\infty) \to [0, +\infty)$,*
- *a certain number $\theta \in [0,1)$,*

such that for $u_0 \in X^{\alpha}$ both estimates

$$\|u(t)\|_Y \leq c(u_0), \quad t > 0 \tag{4}$$

and

$$\|F(u(t))\|_X \leq g(\|u(t)\|_Y)(1 + \|A^{\alpha} u(t)\|_X^{\theta}) \tag{5}$$

$$t \epsilon (0, \tau_{u_0}), \theta \epsilon [\alpha, 1),$$

hold.

To show the existence of global attractor of (1), an important step is to guarantee the compactness of the semigroup. One of the advantages of the parabolic regular initial boundary value problems is that the corresponding semigroup are usually compact. As result from (Hale, 1988) we have that for each $t > 0$, $T(t)$ is a compact map in X^{α}.

Defenition 3.
$A \subset V$ is a global attractor for semigroup $\{T(t)\}$ if A is compact, invariant set and attracts every bounded set of V.

3. Abstract Results

Let Assumption 1 be satiesfied, we are going to establish existence of the global attractor in the present theorem.

Theorem 1.
The solution u of the problem (1) corresponding to $u_0 \,\epsilon\, D(A^{\alpha})$ exist globally for $t \geq 0$ and $T(t)u_0 = u(t,u_0)$ $(t \geq 0)$ defines a strongly continuous semigroup of operator $T(t): D(A^{\alpha}) \to D(A^{\alpha})$, $t \geq 0$. Moreover, when the function c on (4) is bounded on bounded subsets $D(A^{\alpha})$, then $T(t)$ takes bounded sets into bounded sets and compact for $t > 0$. For some $V \subset D(A^{\alpha})$, the priori estimates (5) is asymptotically independent of $u_0 \,\epsilon\, V$, then there a bounded subset of $D(A^{\alpha})$ attracting each point of V.

Proof:
Consider the integral Cauchy formula (2) and write it ain the form

$$u(t) = e^{-At} u_0 + \left(\int_0^{\tau} + \int_{\tau}^{t}\right) e^{-A(t-s)} F(u(s)) ds \tag{6}$$

From the priory estimate (5), for each $u_0 \,\epsilon\, V$, we obtain equation (7) as

$$\|A(t)\|_X \leq \|A^{\alpha}e^{-At}u_0\|_X + \int_0^{\tau}\left\|A^{\alpha}e^{-A(t-s)}\right\|_{L(X,X)} F(\|u(s)\|_X)ds + \int_0^{\tau}\left\|A^{\alpha}e^{-A(t-s)}\right\|_{L(X,X)} g(\|u(s)\|_Y)\left(1+\|A^{\propto}u(s)\|_X^{\theta}\right)ds \qquad (7)$$

With standard properties of analytical semigroup, equation (7) become

$$\|A^{\alpha}(t)\|_X \leq \|A^{\alpha}e^{-At}u_0\|_X + \int_0^{\tau}\left\|A^{\alpha}e^{-A(t-s)}\right\|_{L(X,X)} g(\|u(s)\|_Y)\left(1+\|A^{\propto}u(s)\|_X^{\theta}\right)ds$$

$$\leq c_0\, e^{-at}\|u_0\|_X + g(const.)(1+\sup_{s\epsilon\{\tau,t)}\|u(s)\|_X^{\theta}) \int_0^{t-\tau} c_{\alpha}\frac{e^{-ay}}{y^{\alpha}}\mathrm{dy} \qquad (8)$$

For any $\epsilon > 0$ choose $\sigma(\varepsilon) > t_{u_0}$ for which

$$\sup_{s\epsilon[\sigma(\varepsilon),+\infty)} \|A^{\alpha}(s)\|_X \leq \lim_{t\to+\infty}\|A^{\alpha}(t)\|_X + \varepsilon \qquad (9)$$

Passing equation (8) to the upper limit when $t \to +\infty$, we get for $t > \tau > t_0(u_0)$

$$\sup_{t\to+\infty}\|A^{\alpha}(t)\|_X \leq g(const.)c_{\propto}\frac{\beta(1-\alpha)}{a^{1-\alpha}}\left(1+\sup_{s\epsilon[\sigma(\varepsilon),+\infty)}\|A^{\alpha}(s)\|_X^{\theta}\right) \qquad (10)$$

Making use of the condition (9) and writing

$$C_1 := g(const.)c_{\propto}\frac{\beta(1-\alpha)}{a^{1-\alpha}} \qquad (11)$$

Let us fix some $\tau = \sigma(\varepsilon) > t_0(u_0)$ such that

$$\sup_{s\epsilon[\sigma(\varepsilon),+\infty)}\|A^{\alpha}(s)\|_X^{\theta} \leq 1 + \sup_{t\to+\infty}\|A^{\alpha}(t)\|_X \qquad (12)$$

We obtain from (10) that

$$\lim_{t\to+\infty} sup\|A^{\alpha}(t)\|_X \leq C_1\left(2+\limsup_{t\to+\infty}\|A^{\alpha}(t)\|_X^{\theta})^{\theta}\right) \qquad (13)$$

As seen from (12), the quantity $z := \lim sup_{t\to+\infty}\|A^{\alpha}(t)\|_X$ fulfills the relation

$$z \leq C_1(1+z^{\theta})$$

Which implies that

$$\lim sup_{t\to+\infty}\|A^{\alpha}(t)\|_X \leq z_0 \qquad (14)$$

With z_0 being a positive root of the equation $C_1\left(1+z^{\theta}\right) - z = 0$.
Finally looking back to the defenition of C_1 it is seen that z_0 independent of $u_0 \,\epsilon\, V$. Therefore that condition (14) guarantee that for any $\delta > 0$ and $u_0 \,\epsilon\, V$ there exist a corresponding positive

time in which the solution $u(t, u_0)$ enter an ε-neighborhood of the bounded set $\omega := \{v \epsilon D(A^\alpha); \|A^\alpha v\|_X \le z_0\}$. Thus ω attracts points of *V*, which completes proof Theorema 1. ■

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