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Rainbow Connection Number of Double Quadrilateral Snake Graph

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Abstract

Let graph $G = \big(V(G), E(G)\big)$ be a non trivial connected graph. A graph G with edge coloring is called a rainbow connection, if for every pair of vertices $u, v \in G$ on a path has a different color. The rainbow connection number denoted by $rc(G)$ is the minimum color needed to make graph G rainbow connection. In this study, we will determine the rainbow connection number of double quadrilateral snake graph and alternate double quadrilateral snake graph. The research results show that $rc\big(D(Q_n)\big) = n + 1$ while $rc(AD(Q_3)) = 3,\ if\ \ n = 3,$ and $rc(AD(Q_n)) = n + 1, if\, n\ even\ and\ n \geq 2\ and\ if\ \ n\ odd\ and\ n > 3.$

Keywords: Rainbow Connection Number, Double Quadrilateral Snake Graph, Alternate Double Quadrilateral Snake Graph.

1. INTRODUCTION AND PRELIMINARIES

Graph theory was discovered by a Swiss mathematician named Euler in 1736, which includes the efforts to solve the famous Konigsberg bridge problem in Europe. In general, a graph is a mathematical model that is used to analyze many concrete problems related to the real world [1].

Graph coloring is one of the studies of graphs. One of the developments of graph coloring is the rainbow connection, which was first introduced by Chartrand et al. [3] Rainbow connection is an edge coloring on a non-trivial connected graph with the property that every two vertices can be connected through a rainbow path. Rainbow path is a path from one vertex to another whose edges have different colors. The edge coloring of graph G is called rainbow coloring. If k colors have been used, it is said to be rainbow k coloring. If k is the smallest positive integer of the color needed to make the graph a rainbow connection, then k is called the rainbow connection number denoted by $rc(G)$.

There are several previous studies that have examined rainbow connection numbers, including in 2013, Syafrizal et al. [11] discuss about rainbow connection numbers of fan graph and sun graph. In 2014, Syafrizal et al. [12] discuss about the rainbow connection numbers of the gear graph and circular chain graph. In 2019, Surbakti, N.M., and Sugeng K.A [9] discuss about the rainbow connection number of a watermill graph. In 2019, Maulani et al. [6] discuss about rainbow connection numbers on Graph $C_m \odot P_n$ and Graph $C_m \odot C_n$. In 2019, Parmar et al. [7] discuss about rainbow connection number on graph H and graph $H \odot mk_1$. In 2020, Parmar et al. [8] discuss about rainbow connection number of graphs related to triangular snake graph. Furthermore, in 2021, Suthar et al. [10] discuss about the rainbow connection number of graph $P_n + N_n$, graph $P_n + N_{n-1}$ and quadrilateral snake graph. This research will discuss about the rainbow connection number of double quadrilateral snake graph and alternate double quadrilateral snake graph.

Following are the definitions and theorems used in this research.

Definition 1.1. [2]. Suppose u and v are any distinct vertices on graph G. Then graph G is said to be a connected graph, if there is a path from point u to v.

Definition 1.2. [5]. The distance from u to v in G, denoted by $d(u,v)$ is the length of the shortest path u,v in G.

Definition 1.3. [4]. Graph G is said to be the eccentricity of a point $u \in V(G)$, denoted by $e(u)$ is the farthest distance (maximum shortest path) from u to every point in G, in other words:

$$e(u) = \max\{d(u,v) | u \epsilon V(G)\}$$

Definition 1.4. [4]. The diameter of a graph G denoted by $diam\ (G)$ is the maximum distance between two points in G, namely:

$$diam\ (G) = \max\{e(u) | u \epsilon V(G)\}$$

Definition 1.5. [10]. A quadrilateral snake graph is a graph obtained from a path graph $u_1, u_2, \dots, u_n$, by connecting u_i to vertex v_i and u_{i+1} to vertex $v_i{}'$, as well as connecting vertex v_i to vertex $v_i{}'$, for $1 \leq i < n$. Quadrilateral snake graph is denoted by Q_n, where $n \geq 2$.

Definition 1.6. [10]. An alternate quadrilateral snake graph is a graph obtained from a path graph $u_1, u_2, \dots, u_n$, by connecting u_j to vertex v_j and u_{j+1} to vertex $v_j{}'$, as well as connecting vertex v_j to vertex $v_j{}'$, $j = 2i - 1, i = 1,2, \dots, \left|\frac{n}{2}\right|$. Alternate quadrilateral snake graph is denoted by $\left(A(Q_n)\right)$, where $n \geq 2$.

Definition 1.7. [10]. A double quadrilateral snake graph is a graph that consists of two quadrilateral snake graphs that have the same path. Double quadrilateral snake graph is denoted by $D(Q_n)$, where $n \geq 2$.

Definition 1.8. [10]. An alternate double quadrilateral snake graph is a graph that consists of two alternate snake graphs that have the same path. Alternate double quadrilateral snake graph is denoted by $AD(Q_n)$, where $n \geq 2$.

Theorem 1.1. [3]. *Let G be a non-trivial connected graph, then $rc(G) \geq diam(G)$.*

2. MAIN RESULTS

2.1 Rainbow Connection Number of Double Quadrilateral Snake Graph

Before discussing the theorem of $rc(D(Q_n))$, we will first determine the pattern of the rainbow connection number presented in Table 2.1.

Table 2.1. Common Patterns of Rainbow Connection Number of Double Quadrilateral Snake Graph

n	Graph	Rainbow Connection Number
2	$D(Q_2)$	$rc(D(Q_2)) = 3$
3	$D(Q_3)$	$rc(D(Q_3)) = 4$
4	$D(Q_4)$	$rc(D(Q_4)) = 5$
5	$D(Q_5)$	$rc(D(Q_5)) = 6$
6	$D(Q_6)$	$rc(D(Q_6)) = 7$
$\vdots$	$\vdots$	$\vdots$
n	$D(Q_n)$	$rc(D(Q_n)) = n + 1$

Based on table 2.1, then theorem 2.1 is obtained.

Theorem 2.1. *If $D(Q_n)$ is a double quadrilateral snake graph, then rainbow connection number of $D(Q_n)$, where $n \geq 2$ is $rc(D(Q_n)) = n + 1$.*

Proof. Let $D(Q_n)$, with set of vertices and set of edges, as follows: $V((D(Q_n)) = \{u_i | i \in \{1,2,\dots,n\}\} \cup \{v_i | i \in \{1,2,\dots,n-1\}\} \cup \{v_i' | \in i\{1,2,..,n-1\}\} \cup \{w_i | \in i\{1,2,\dots,n-1\}\} \cup \{w_i' | \in i\{1,2,\dots n-1\}\}$, and $E(D(Q_n)) = \{u_i u_{i+1} | i \in \{1,2,\dots,n\}\} \cup \{u_i v_i | i \in \{1,2,\dots,n-1\}\} \cup \{u_{i+1} v_i' | i \in \{1,2,\dots,n-1\}\} \cup \{v_i v_i' | i \in \{1,2,\dots n-1\}\} \cup \{u_i w_i | i \in \{1,2,..,n-1\} \cup \{u_{i+1} w_i' | i \in \{1,2,\dots,n-1\}\} \cup \{w_i w_i' | i \in \{1,2,\dots,n-1\}\}$

Firstly, we will show that the lower bound of $D(Q_n)$, we know that $diam(D(Q_n)) = n + 1$, therefore, based on Theorem 1.1 then $rc(D(Q_n)) \geq n + 1$.

Secondly, we will show that the upper bound of $D(Q_n)$, with edge coloring function $f: E\left(D(Q_n)\right) \rightarrow \{1,2,\dots,n+1\}$ as follows:

$$f(e) = \begin{cases} i, & e = (u_i, u_{i+1}); \quad for\ 1 \leq i \leq n-1 \\ i, & e = (v_i, v_i'); \quad for\ 1 \leq i \leq n-1 \\ i, & e = (w_i, w_i'); \quad for\ 1 \leq i \leq n-1 \\ n, & e = (u_i, v_i); \quad for\ 1 \leq i \leq n-1 \\ n+1, & e = (u_{i+1}, v_i'); \quad for\ 1 \leq i \leq n-1 \\ n+1, & e = (u_i, w_i); \quad for\ i = 1 \\ i, & e = (u_i, w_i); \quad for\ i = n-1 \\ 2+i, & e = (u_{i+1}, w_{i+1}); \quad for\ 1 \leq i \leq n-1 \\ 1, & e = (u_{i+1}, w_i') \quad for\ 1 \leq i \leq n-1 \\ i, & e = (u_{i+1}, w_i'); \quad for\ i = n-1 \end{cases}$$

It will be shown that for every two vertices u and v in $D(Q_n)$, there is some rainbow path u, v ,which can be seen in Table 2.2:

Table 2.2. Rainbow path u, v of $D(Q_n)$

Case	Vertex u	Vertex v	Condition	Rainbow Path
1	u_i	u_j	$1 \le i \le n$, $1 \le j \le n$	$u_i, u_{i+1}, \dots, u_j, i < j$ $u_i, u_{i-1}, \dots, u_j, i > j$
2	v_i	v_j	$1 \le i \le n-1$, $1 \le j \le n-1$	$v_i, v_i', u_{i+1}, \dots, u_j, v_j, i < j$ $v_i, u_i, u_{i-1}, \dots, u_{j+1}, v_j', v_j, i > j$
3	$v_i{}'$	$v_j{}'$	$1 \le i \le n-1$, $1 \le j \le n-1$	$v_i', u_{i+1}, \dots, u_j, v_j, v_j', i < j$ $v_i', v_i, u_i, u_{i-1}, \dots u_{j+1}, v_j', i > j$
4	u_i	v_j	$1 \le i \le n$, $1 \le j \le n-1$	$u_i, u_{i+1}, \dots, u_j, v_j, i < j$ $u_i, u_{i-1}, \dots, u_j, v_j, i > j$ $u_i, v_j, i = j$
5	u_i	$v_j{}'$	$1 \le i \le n$, $1 \le j \le n-1$	$u_i, u_{i+1}, \dots, u_j, v_j, v_j', i < j$ $u_i, u_{i-1}, \dots, u_{j+1}, v_j{}', i > j$ $u_i, v_j, v_j', i = j$
6	v_i	$v_j{}'$	$1 \le i \le n-1$, $1 \le j \le n-1$	$v_i, v_i', u_{i+1}, \dots, u_j, v_j, v_j', i < j$ $v_i, u_i, u_{i-1}, \dots, u_{j+1}, v_j', i > j$ $v_i, v_j', i = j$
7	w_i	w_j	$1 \le i \le n-1$, $1 \le j \le n-1$	$w_i, u_i, u_{i+1}, \dots, u_j, w_j, i < j$ $w_i, u_i, u_{i-1}, \dots, u_j, w_j, i > j$
8	$w_i{}'$	$w_j{}'$	$1 \le i \le n-2$, $1 \le j \le n-2$	$w_i', u_{i+1}, \dots, u_j, w_j, w_j', i < j$ $w_i', w_i, u_i, u_{i-1}, \dots, u_{j+1}, w_j', i > j$
			$1 \le i \le n-2$ $j = n-1$	$w_i', u_{i+1}, \dots, u_j, w_j', i < j$
			$i = n-1$ $1 \le j \le n-2$	$w_i', u_i, u_{i-1}, \dots, u_{j+1}, w_j', i > j$
9	u_i	w_j	$1 \le i \le n$, $1 \le j \le n-1$	$u_i, u_{i+1}, \dots, u_j, w_j, i < j$ $u_i, u_{i-1}, \dots, u_j, w_j, i > j$ $u_i, w_j, i = j$
10	u_i	$w_j{}'$	$1 \le i \le n$, $1 \le j \le n-2$	$u_i, u_{i+1}, \dots, u_j, w_j, w_j', i < j$ $u_i, u_{i-1}, \dots, u_{j+1}, w_j{}', i > j$ $u_i, w_j, w_j', i = j$
			$1 \le i \le n$, $j = n-1$	$u_i, u_{i+1}, \dots, u_j, u_{j+1}, w_j', i < j$ $u_i, w_j{}', i > j$ $u_i, u_{j+1}, w_j', i = j$
11	w_i	$w_j{}'$	$i = 1$, $1 \le j \le n-2$	$w_i, u_i, u_{i+1}, \dots, u_j, w_j, w_j{}', i < j$ $w_i, w_j', i = j = 1$
			$2 \le i \le n-1$, $j = 1$	$w_i, u_i, u_{i-1}, \dots, u_{j+1}, w_j', i > j$
			$i = 1, j = n-1$	$w_i, u_i, u_{i+1}, \dots, u_{j+1}, w_j{}', i < j$
			$2 \le i \le n-1$, $j = n-1$	$w_i, w_i{}', u_{i+1}, \dots, u_{j+1}, w_j{}', i < j$

				$w_i, w_j', i = j = n-1$
			$2 \le i \le n-1$,	$w_i, w_i', u_{i+1}, \dots, u_j, w_j, w_j', i < j$
			$2 \le j \le n-2$	$w_i, u_i, u_{i-1}, \dots, u_{j+1}, w_j', i > j$
				$w_i, w_j', i = j$
12	v_i	w_j	$1 \le i \le n-1$,	$v_i, v_i', u_{i+1}, \dots, u_j, w_j, i < j$
			$1 \le j \le n-1$	$v_i, u_i, u_{i-1}, \dots, u_{j+1}, w_j, i > j$
				$v_i, u_i, w_j, i = j$
13	v_i	w_j'	$1 \le i \le n-1$,	$v_i, v_i', u_{i+1}, \dots, u_j, w_j, w_j', i < j$
			$1 \le j \le n-2$	$v_i, u_i, u_{i-1}, \dots, u_{j+1}, w_j', i > j$
				$v_i, u_i, w_j, w_j', i = j = 1$
				$v_i, v_i', u_{i+1}, w_j', i = j \ge 2$
			$1 \le i \le n-1$,	$v_i, v_i', u_{i+1}, \dots, u_{j+1}, w_j', i < j$
			$j = n-1$	$v_i, v_i', u_{i+1}, w_j', i = j = n+1$
14	v_i'	w_j	$1 \le i \le n-1$,	$v_i', u_{i+1}, \dots, u_j, w_j, i < j$
			$2 \le j \le n-1$	$v_i', v_i, u_i, u_{i-1}, \dots, u_{j+1}, w_j', w_j, i > j$
				$v_i', u_{i+1}, w_j', w_j, i = j \ge 2$
			$1 \le i \le n-1$,	$v_i', u_{i+1}, \dots, u_j, w_j, i < j$
			$j = 1$	$v_i', v_i, u_i, u_{i-1}, \dots, u_j, w_j, i > j$
				$v_i', v_i, u_i, w_j, i = j = 1$
15	v_i'	w_j'	$1 \le i \le n-1$,	$v_i', u_{i+1}, \dots, u_{j+1}, w_j', i < j$
			$1 \le j \le n-1$	$v_i', v_i, u_i, u_{i-1}, \dots, u_j, w_j', i > j$
				$v_i, u_{i+1}, w_j', i = j$

Because, every two vertices u and v in $D(Q_n)$, there exists some rainbow path u, v, then $rc(D(Q_n)) \le n+1$. Therefore, based on the lower bound and upper bound of $D(Q_n)$, it is proven that $rc(D(Q_n)) = n+1$.

Figure 2.1 is an example double quadrilateral snake graph, where $n = 6$. It can be seen that if $n = 6$, then $rc(D(Q_6)) = 7$.

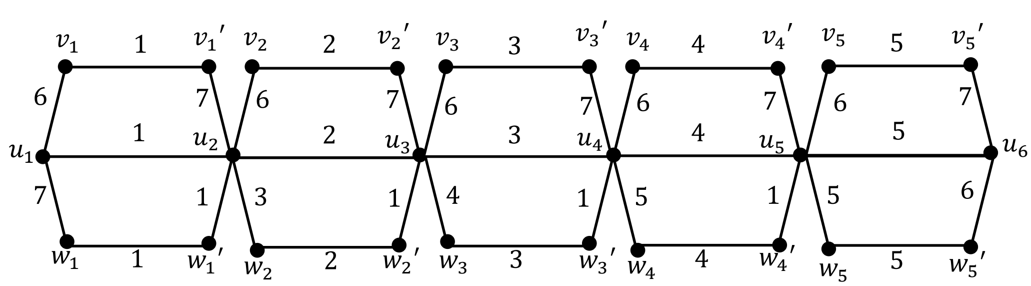

Figure 2.1. $D(Q_6)$, rainbow 7-coloring

2.2 Rainbow Connection Number of Alternate Double Quadrilateral Snake Graph

Before discussing the theorem of rainbow connection number of $AD(Q_n)$, we will first determine the pattern of the rainbow connection number of $(AD(Q_n))$ presented in Table 2.3:

Table 2.3. Common Pattern of Rainbow Connection Number of Alternate Double Quadrilateral Snake Graph

n	Graph	Rainbow Connection Number
2	$AD(Q_2)$	$rc\big(AD(Q_2)\big) = 3$
3	$AD(Q_3)$	$rc\big(AD(Q_3)\big) = 3$
4	$AD(Q_4)$	$rc\big(AD(Q_4)\big) = 5$
5	$AD(Q_5)$	$rc\big(AD(Q_5)\big) = 6$
6	$AD(Q_6)$	$rc\big(AD(Q_6)\big) = 7$
7		
⋮	⋮	⋮
n	$AD(Q_n)$	$rc(AD(Q_n)) = \begin{cases} 3, & if\ \ n = 3 \\ n+1, & if\ n\ is\ even\ and\ n \geq 2\ and\ if\ n\ is\ odd\ and\ n > 3 \end{cases}$

Based on table 2.3, then theorem 2.2 is obtained.

Theorem 2.2. *If $AD(Q_n)$ is an alternate double quadrilateral snake graph, then rainbow connection number of $AD(Q_n)$, where $n \geq 2$ is defined as*

$$rc(AD(Q_n)) = \begin{cases} 3, & if\ n = 3 \\ n+1, & if\ n\ is\ even\ and\ n \geq 2\ and\ if\ n\ is\ odd\ and\ n > 3 \end{cases}$$

Proof. Let $AD(Q_n)$, for odd numbers of n with a set of vertices and a set of edges as follows: $V((AD(Q_n)) = \{u_i | i \in \{1,2,\dots,n\}\} \cup \{v_i | i \in \{1,3,\dots,n-2\}\} \cup \{v_i' | \in i\{1,3,.,n-2\}\} \cup \{w_i | \in i\{1,3,\dots,n-2\}\} \cup \{w_i' | \in i\{1,3,\dots n-2\}\}$ and $E\big(AD(Q_n)\big) = \{u_i u_{i+1} | i \in \{1,2,\dots,n\}\} \cup \{u_i v_i | i \in \{1,3,.,n-2\}\} \cup \{u_{i+1} v_i' | i \in \{1,3,.,n-2\}\} \cup \{v_i v_i' | i \in \{1,3,\dots n-2\}\} \cup \{u_i w_i | i \in \{1,3,..,n-2\} \cup \{u_{i+1} w_i' | i \in \{1,3,\dots,n-2\}\} \cup \{w_i w_i' | i \in \{1,3,\dots,n-2\}\}$

While $AD(Q_n)$, for even numbers of n with a set of vertices and a set of edges, as follows:

$V((AD(Q_n)) = \{u_i | i \in \{1,2,\dots,n\}\} \cup \{v_i | i \in \{1,3,\dots,n-1\}\} \cup \{v_i' | \in i\{1,3,.,n-1\}\} \cup \{w_i | \in i\{1,3,\dots,n-1\}\} \cup \{w_i' | \in i\{1,3,\dots n-1\}\}$ and $E\big(AD(Q_n)\big) = \{u_i u_{i+1} | i \in \{1,2,\dots,n\}\} \cup \{u_i v_i | i \in \{1,3,.,n-1\}\} \cup \{u_{i+1} v_i' | i \in \{1,3,.,n-1\}\} \cup \{v_i v_i' | i \in \{1,3,\dots n-1\}\} \cup \{u_i w_i | i \in \{1,3,..,n-1\} \cup \{u_{i+1} w_i' | i \in \{1,3,\dots,n-1\}\} \cup \{w_i w_i' | i \in \{1,3,\dots,n-1\}\}$.

Case 1: $if\, n = 3$

Firstly, we will show the lower bound of $AD(Q_3)$, we know that $m(AD(Q_3)) = 3$, therefore, based on Theorem 1.1 then $rc(AD(Q_3)) \geq 3$.

Secondly, we will show the upper bound of $AD(Q_3)$, with edge coloring function $f\!: E\ (AD(Q_3)) \to \{1,2,3\}$ as follows:

$$f(e)=\begin{cases} i, & e=(u_i,u_{i+1}); \quad for\ 1\le i\le 3\\ 1, & e=(v_i,v_i'); \quad for\ i=1\\ 1, & e=(w_i,w_i'); \quad for\ i=1\\ 3, & e=(u_i,v_i); \quad for\ i=3\\ 3, & e=(u_{i+1},v_i'); \quad for\ i=3\\ 3, & e=(u_i,w_i); \quad for\ i=3\\ 1, & e=(u_{i+1},w_i'); \quad for\ i=1\\ 1, & e=(u_{i+1},w_i); \quad for\ i=1 \end{cases}$$

It will be shown that for every two vertices u and v in $AD(Q_3)$, there is some rainbow path u,v ,which can be seen in Table 2.4:

Table 2.4. Rainbow path u,v of $AD(Q_3)$

Case	Vertex u	Vertex v	Condition	Rainbow Path
1	u_i	u_j	$1\le i\le 3, 1\le j\le 3$	$u_i,u_{i+1},\dots,u_j, i<j$ $u_i,u_{i-1},\dots,u_j, i>j$
2	u_i	v_j	$i=j=1$	$u_i,v_j, i=j$
			$2\le i\le 3,\ j=1$	$u_i,u_{i-1},v_j, i>j$
3	u_i	$v_j{}'$	$i=j=1$	$u_i,u_{j+1},v_j', i=j$
			$i=2, j=1$	$u_i,v_j', i>j$
			$i=3, j=1$	$u_i,u_{i-1},v_j', i>j$
4	v_i	$v_j{}'$	$i=j=1$	$v_i,v_j', i=j$
5	u_i	w_j	$1\le i\le 3,$ $j=1$	$u_i,u_{i-1},u_j,w_j, i>j$ $u_i,w_j, i=j$
	u_i	$w_j{}'$	$i=j=1$	$u_i,w_j,w_j', i=j$
			$i=2,\ j=1$	$u_i,w_j', i>j$
			$i=3, j=1$	$u_i,u_{j+1},w_j', i>j$
7	w_i	$w_j{}'$	$i=j=1$	$w_i,w_j', i=j$
8	v_i	w_j	$i=j=1$	$v_i,u_i,w_j, i=j$
9	v_i	$w_j{}'$	$i=j=1$	$v_i,u_i,w_j,w_j', i=j$
10	$v_i{}'$	w_j	$i=j=1$	$v_i{}',v_i,u_i,w_j, i=j$
11	$v_i{}'$	$w_j{}'$	$i=j=1$	$v_i,u_{i+1},w_j', i=j$

Because, every two vertices u and v in $AD(Q_3)$, there exists some rainbow path u,v, then $rc(AD(Q_3))\le 3$. Therefore, based on the lower bound and upper bound of $AD(Q_3)$, it is proven that $rc(AD(Q_3))=3$.

Figure 2.2 is an example alternate double quadrilateral snake graph, where $n=3$. It can be seen that if $n=3$, then $rc(AD(Q_3))=3$.

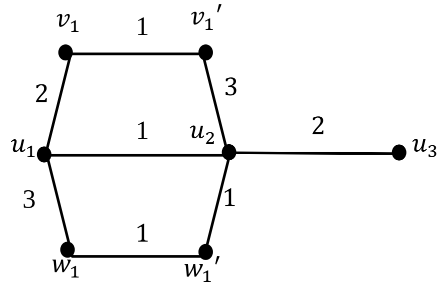

Figure 2.2. $AD(Q_3)$, rainbow 3-coloring

Case 2. $if\ n\ is\ even\ and\ n \geq 2\ and\ if\ n\ is\ odd\ and\ n > 3$

In proving case 2 it will consist of 2 subcases, as follows:

Subcase 2.1. *If n is even and* $n \geq 2$

Firstly, we will show that the lower bound of $AD(Q_n)$, if n is even, we know that $diam(AD(Q_n)) = n + 1$, therefore, based on Theorem 1.1, then $rc(AD(Q_n)) \geq n + 1$.

Secondly, we will show the upper bound of $AD(Q_n)$, with edge coloring function $f: E\left(AD(Q_n)\right) \to \{1,2, \dots, n + 1\}$ as follows:

$$f(e) = \begin{cases} i, & e = (u_i, u_{i+1}); \quad for\ 1 \leq i \leq n-1 \\ i, & e = (v_i, v_i{}'); \quad for\ i \in \{1,3,5, \dots n-1\} \\ i, & e = (w_i, w_i{}'); \quad for\ i \in \{1,3,5, \dots n-1\} \\ n, & e = (u_i, v_i); \quad for\ i \in \{1,3,5, \dots n-1\} \\ n+1, & e = (u_{i+1}, v_i{}'); \quad for\ i \in \{1,3,5, \dots n-1\} \\ n+1, & e = (u_i, w_i); \quad for\ i = 1 \\ i, & e = (u_i, w_i); \quad for\ i = n-1 \\ i+1, & e = (u_i, w_i); \quad for\ i \in \{1,3,5, \dots n-1\} \\ 1, & e = (u_{i+1}, w_i{}'); \quad for\ i \in \{1,3,5, \dots n-1\} \\ i, & e = (u_{i+1}, w_i{}'); \quad for\ i = n-1 \end{cases}$$

It will be shown that for every two vertices u and v in $AD(Q_n)$, there is some rainbow path u, v ,which can be seen in Table 2.5:

Table 2.5. Rainbow path u, v of $AD(Q_n)$

Case	Vertex u	Vertex v	Condition	Rainbow Path
1	u_i	u_j	$1 \leq i \leq n$, $1 \leq j \leq n$	$u_i, u_{i+1}, \dots, u_j,\ i < j$
				$u_i, u_{i-1}, \dots, u_j, i > j$
2	v_i	v_j	$i = 1,3,5, \dots, n-1$	$v_i, v_i', u_{i+1}, \dots, u_j, v_j, i < j$
			$j = 1,3,5, \dots, n-1$	$v_i, u_i, u_{i-1}, \dots, u_{j+1}, v_j', v_j, i > j$
3	$v_i{}'$	$v_j{}'$	$i = 1,3,5, \dots, n-1$	$v_i', u_{i+1}, \dots, u_j, v_j, v_j', i < j$
			$j = 1,3,5, \dots, n-1$	$v_i', v_i, u_i, u_{i-1}, \dots u_{j+1}, v_j', i > j$
4	u_i	v_j	$1 \leq i \leq n$,	$u_i, u_{i+1}, \dots, u_j, v_j, i < j$
			$j = 1,3,5, \dots, n-1$	$u_i, u_{i-1}, \dots, u_j, v_j, i > j$
				$u_i, v_j, i = j$

5	u_i	$v_j{}'$	$1 \le i \le n,$ $j = 1,3,5,\dots,n-1$	$u_i, u_{i+1}, \dots, u_j, v_j, v_j', i < j$ $u_i, u_{i-1}, \dots, u_{j+1}, v_j{}', i > j$ $u_i, v_j, v_j', i = j$
6	v_i	$v_j{}'$	$i = 1,3,5,\dots,n-1$ $j = 1,3,5,\dots,n-1$	$v_i, v_i', u_{i+1}, \dots, u_j, v_j, v_j', i < j$ $v_i, u_i, u_{i-1}, \dots, u_{j+1}, v_j', i > j$ $v_i, v_j', i = j$
7	w_i	w_j	$i = 1,3,5,\dots,n-1$ $j = 1,3,5,\dots,n-1$	$w_i, u_i, u_{i+1}, \dots, u_j, w_j, i < j$ $w_i, u_i, u_{i-1}, \dots, u_j, w_j, i > j$
8	$w_i{}'$	$w_j{}'$	$i = 1,3,5,\dots,n-3,$ $j = 1,3,5,\dots,n-3,$	$w_i', u_{i+1}, \dots, u_j, w_j, w_j', i < j$ $w_i', w_i, u_i, u_{i-1}, \dots, u_{j+1}, w_j', i > j$
			$i = 1,3,5,\dots,n-3,$ $j = n-1$	$w_i', u_{i+1}, \dots, u_j, w_j', i < j$
			$i = n-1$ $j = 1,3,5,\dots,n-3,$	$w_i', u_i, u_{i-1}, \dots, u_{j+1}, w_j', i > j$
9	u_i	w_j	$1 \le i \le n,$ $j = 1,3,5,\dots,n-1$	$u_i, u_{i+1}, \dots, u_j, w_j, i < j$ $u_i, u_{i-1}, \dots, u_j, w_j, i > j$ $u_i, w_j, i = j$
10	u_i	$w_j{}'$	$1 \le i \le n,$ $j = 1,3,5,\dots,n-3,$	$u_i, u_{i+1}, \dots, u_j, w_j, w_j', i < j$ $u_i, u_{i-1}, \dots, u_{j+1}, w_j{}', i > j$ $u_i, w_j, w_j', i = j$
			$1 \le i \le n,$ $j = n-1$	$u_i, u_{i+1}, \dots, u_j, u_{j+1}, w_j', i < j$ $u_i, w_j{}', i > j$ $u_i, u_{j+1}, w_j', i = j$
11	w_i	$w_j{}'$	$i = 1,$ $j = 1,3,5,\dots,n-3$	$w_i, u_i, u_{i+1}, \dots, u_j, w_j, w_j{}', i < j$ $w_i, w_j', i = j = 1$
			$i = 3,5,\dots,n-1,$ $j = 1$	$w_i, u_i, u_{i-1}, \dots, u_{j+1}, w_j{}', i > j$
			$i = 1, j = n-1$	$w_i, u_i, u_{i+1}, \dots, u_{j+1}, w_j{}', i < j$
			$i = 3,5,\dots,n-1$ $j = n-1$	$w_i, w_i{}', u_{i+1}, \dots, u_{j+1}, w_j{}', i < j$ $w_i, w_j', i = j = n-1$
			$i = 3,5,\dots,n-1$ $j = 3,5,\dots,n-3$	$w_i, w_i', u_{i+1}, \dots, u_j, w_j, w_j{}', i < j$ $w_i, u_i, u_{i-1}, \dots, u_{j+1}, w_j{}', i > j$ $w_i, w_j', i = j$
12	v_i	w_j	$i = 1,3,5,\dots,n-1$ $j = 1,3,5,\dots,n-1$	$v_i, v_i', u_{i+1}, \dots, u_j, w_j, i < j$ $v_i, u_i, u_{i-1}, \dots, u_{j+1}, w_j, i > j$ $v_i, u_i, w_j, i = j$
13	v_i	$w_j{}'$	$i = 1,3,5,\dots,n-1$ $j = 1,3,5,\dots,n-3$	$v_i, v_i{}', u_{i+1}, \dots, u_j, w_j, w_j', i < j$ $v_i, u_i, u_{i-1}, \dots, u_{j+1}, w_j', i > j$ $v_i, u_i, w_j, w_j', i = j = 1$
			$i = 1,3,5,\dots,n-1$ $j = n-1$	$v_i, v_i{}', u_{i+1}, \dots, u_{j+1}, w_j', i < j$ $v_i, v_i', u_{i+1}, w_j', i = j = n+1$
14	$v_i{}'$	w_j	$i = 1,3,5,\dots,n-1$ $j = 3,5,\dots,n-1$	$v_i', u_{i+1}, \dots, u_j, w_j, i < j$ $v_i', v_i, u_i, u_{i-1},, u_{j+1}, w_j', w_j, i > j$ $v_i', u_{i+1}, w_j', w_j, i = j \ge 3$

15	v_i'	w_j'	$i = 1,3,5, \dots, n-1$ $j = 1$ $i = 1,3,5, \dots, n-1$ $j = 1,3,5, \dots, n-1$	$v_i', v_i, u_i, u_{i-1}, \dots, u_j, w_j, i > j$ $v_i', v_i, u_i, w_j, i = j = 1$ $v_i', u_{i+1}, \dots, u_{j+1}, w_j', i < j$ $v_i', v_i, u_i, u_{i-1}, \dots, u_j, w_j', i > j$ $v_i, u_{i+1}, w_j', i = j$

Because, for every pair of vertices $u, v \in AD(Q_n)$, on a path has a different color, then $rc(AD(Q_n)) \leq n + 1$. Thus, based on the lower bound and upper bound of $AD(Q_n)$, it is proven that $rc(AD(Q_n)) = n + 1$.

Figure 2.3 is an example of rainbow connection number of $AD(Q_n)$, where $n = 6$. It can be seen that if $n = 6$, then $rc(AD(Q_6)) = 7$.

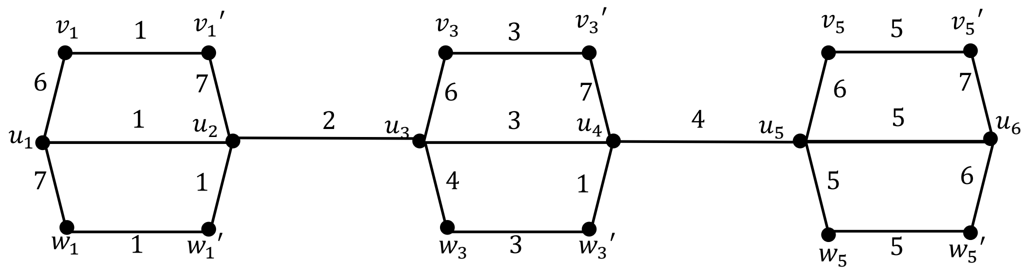

Figure 2.3 $AD(Q_6)$, rainbow 7-coloring

Subcase 2.2. *if* n *is odd and* $n > 3$

Firstly, we will show the lower bound of the $AD(Q_n)$, if n is odd, we know that $diam(AD(Q_n)) = n$, therefore, based on Theorem 1.1 then $rc\big(AD(Q_n)\big) \geq n$. Let $AD(Q_5)$, suppose there is a rainbow coloring on $AD(Q_5)$, with a set of colors $\{1,2,3,4,5\}$. We define rainbow 5-coloring function, $c{:}E\ (AD(Q_5)) \to \{1,2,3,4,5\}$, as follows:

$$c(e) = \begin{cases} 1, & e = \{(u_1, u_2), (v_1, v_1'), (w_1, w_1'), (u_2, w_1')\} \\ 2, & e = \{(u_2, u_3)\} \\ 3, & e = \{(u_3, u_4), (v_3, v_3'), (w_3, w_3'), (u_3, w_3)\} \\ 4, & e = \{(u_1, v_1), (u_3, v_3), (u_4, w_3'), (u_4, u_5)\} \\ 5, & e = \{(u_2, v_1'), (u_4, v_3'), (u_1, w_1)\} \end{cases}$$

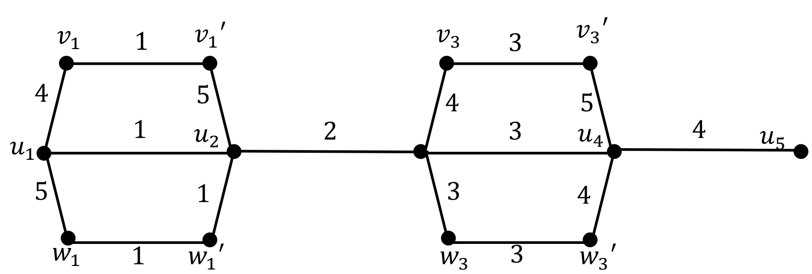

Figure 2.4 $AD(Q_5)$, not rainbow 5-coloring

Based on the definition of the coloring, note that the edges (u_4, u_5) cannot be colored by 1,2,3,4 or 5. Suppose the edges are colored 4, then there will be a path that is not a rainbow path, namely path (u_5, u_4, w'_3). Suppose the edges (u_4, w'_3) are colored 5, then there will be a path that is not a rainbow path, namely path $(w_1, u_1, u_2, u_3, u_4, u_5)$. Suppose the edges (u_4, w'_3) are colored 1,2 or 3, there will be path that is not a rainbow path, namely path $(u_1, u_2, u_3, u_4, u_5)$, because for $AD(Q_5)$ rainbow 5-coloring does not apply, then there is a contradiction with the assumption above. Therefore, we get $rc(AD(Q_5)) > 6$ or $rc(AD(Q_5)) \geq 6$.

Secondly, we will show the upper bound of $AD(Q_n)$, with edge coloring function $f: E\left(AD(Q_n)\right) \to \{1,2,\ldots,n+1\}$, as follows:

$$f(e) = \begin{cases} i, & e = (u_i, u_{i+1}); & for\ 1 \leq i \leq n-1 \\ i, & e = (v_i, {v_i}'); & for\ i \in \{1,3,5,\ldots n-2\} \\ i, & e = (w_i, {w_i}'); & for\ i \in \{1,3,5,\ldots n-2\} \\ n, & e = (u_i, v_i); & for\ i \in \{1,3,5,\ldots n-2\} \\ n+1, & e = (u_{i+1}, {v_i}'); & for\ i \in \{1,3,5,\ldots n-2\} \\ n+1, & e = (u_i, w_i); & for\ i = 1 \\ i+1, & e = (u_i, w_i); & for\ i \in \{1,3,5,\ldots n-2\} \\ 1, & e = (u_{i+1}, {w_i}'); & for\ i \in \{1,3,5,\ldots n-2\} \\ i+2, & e = (u_{i+1}, {w_i}'); & for\ i = n-2 \end{cases}$$

It will be shown that for every two vertices u and v in $AD(Q_n)$, there is some rainbow path u, v ,which can be seen in Table 2.6:

Table 2.6. Rainbow path u, v of $AD(Q_n)$

Case	Vertex u	Vertex v	Condition	Rainbow Path
1	u_i	u_j	$1 \leq i \leq n$, $1 \leq j \leq n$	$u_i, u_{i+1}, \ldots, u_j, i < j$ $u_i, u_{i-1}, \ldots, u_j, i > j$
2	v_i	v_j	$i = 1,3,5,\ldots,n-2$ $j = 1,3,5,\ldots,n-2$	$v_i, v'_i, u_{i+1}, \ldots, u_j, v_j, i < j$ $v_i, u_i, u_{i-1}, \ldots, u_{j+1}, v'_j, v_j, i > j$
3	${v_i}'$	${v_j}'$	$i = 1,3,5,\ldots,n-2$ $j = 1,3,5,\ldots,n-2$	$v'_i, u_{i+1}, \ldots, u_j, v_j, v'_j, i < j$ $v'_i, v_i, u_i, u_{i-1}, \ldots u_{j+1}, v'_j, i > j$
4	u_i	v_j	$1 \leq i \leq n$, $j = 1,3,5,\ldots,n-2$	$u_i, u_{i+1}, \ldots, u_j, v_j, i < j$ $u_i, u_{i-1}, \ldots, u_j, v_j, i > j$ $u_i, v_j, i = j$
5	u_i	${v_j}'$	$1 \leq i \leq n$, $j = 1,3,5,\ldots,n-2$	$u_i, u_{i+1}, \ldots, u_j, v_j, v'_j, i < j$ $u_i, u_{i-1}, \ldots, u_{j+1}, {v_j}', i > j$ $u_i, v_j, v'_j, i = j$
6	v_i	${v_j}'$	$i = 1,3,5,\ldots,n-2$ $j = 1,3,5,\ldots,n-2$	$v_i, v'_i, u_{i+1}, \ldots, u_j, v_j, v'_j, i < j$ $v_i, u_i, u_{i-1}, \ldots, u_{j+1}, v'_j, i > j$ $v_i, v'_j, i = j$
7	w_i	w_j	$i = 1,3,5,\ldots,n-2$ $j = 1,3,5,\ldots,n-2$	$w_i, u_i, u_{i+1}, \ldots, u_j, w_j, i < j$ $w_i, u_i, u_{i-1}, \ldots, u_j, w_j, i > j$
8	${w_i}'$	${w_j}'$	$i = 1,3,5,\ldots,n-2$, $j = 1,3,5,\ldots,n-2$	$w'_i, u_{i+1}, \ldots, u_j, w_j, w'_j, i < j$ $w'_i, w_i, u_i, u_{i-1}, \ldots, u_{j+1}, w'_j, i > j$

9	u_i	w_j	$1 \leq i \leq n$, $j = 1,3,5, \dots, n-2$	$u_i, u_{i+1}, \dots, u_j, w_j, i < j$ $u_i, u_{i-1}, \dots, u_j, w_j, i > j$ $u_i, w_j, i = j$
10	u_i	$w_j{}'$	$1 \leq i \leq n$, $j = 1,3,5, \dots, n-2$	$u_i, u_{i+1}, \dots, u_j, w_j, w_j', i < j$ $u_i, u_{i-1}, \dots, u_{j+1}, w_j{}', i > j$ $u_i, w_j, w_j', i = j$
11	w_i	$w_j{}'$	$i = 1$, $j = 1,3,5, \dots, n-2$	$w_i, u_i, u_{i+1}, \dots, u_j, w_j, w_j{}', i < j$ $w_i, w_j', i = j = 1$
			$i = 3,5, \dots, n-2$, $j = 1$	$w_i, u_i, u_{i-1}, \dots, u_{j+1}, w_j{}', i > j$
			$i = 3,5, \dots, n-2$ $j = 3,5, \dots, n-2$	$w_i, w_i', u_{i+1}, \dots, u_j, w_j, w_j{}', i < j$ $w_i, u_i, u_{i-1}, \dots, u_{j+1}, w_j{}', i > j$ $w_i, w_j', i = j$
12	v_i	w_j	$i = 1,3,5, \dots, n-2$ $j = 1,3,5, \dots, n-2$	$v_i, v_i', u_{i+1}, \dots, u_j, w_j, i < j$ $v_i, u_i, u_{i-1}, \dots, u_{j+1}, w_j, i > j$ $v_i, u_i, w_j, i = j$
13	v_i	$w_j{}'$	$i = 1,3,5, \dots, n-2$ $j = 1,3,5, \dots, n-4$	$v_i, v_i{}', u_{i+1}, \dots, u_j, w_j, w_j', i < j$ $v_i, u_i, u_{i-1}, \dots, u_{j+1}, w_j', i > j$ $v_i, u_i, w_j, w_j', i = j$
			$i = 1,3,5, \dots, n-2$ $j = n-2$	$v_i, v_i{}', u_{i+1}, \dots, u_{j+1}, w_j', i < j$ $v_i, v_i', u_{i+1}, w_j', i = j$
14	$v_i{}'$	w_j	$i = 1,3,5, \dots, n-2$ $j = 1,3,5, \dots, n-2$	$v_i', u_{i+1}, \dots, u_j, w_j, i < j$ $v_i', v_i, u_i, u_{i-1}, \dots, u_{j+1}, w_j', w_j, i > j$ $v_i{}', v_i, u_i, w_j, i = j = 1$
15	$v_i{}'$	$w_j{}'$	$i = 1,3,5, \dots, n-2$ $j = 1,3,5, \dots, n-2$	$v_i{}', u_{i+1}, \dots, u_{j+1}, w_j', i < j$ $v_i{}', v_i, u_i, u_{i-1}, \dots, u_j, w_j', i > j$ $v_i, u_{i+1}, w_j', i = j$

Because, every two vertices u and v in $AD(Q_n)$, there exists some rainbow path u, v, then $rc(AD(Q_n)) \leq n + 1$. Therefore, based on the lower bound and upper bound of $AD(Q_n)$, it is proven that $rc(AD(Q_n)) = n + 1$.

Figure 2.5 is an example alternate double quadrilateral snake graph, where $n = 7$. It can be seen that if $n = 7$, then $rc(AD(Q_7)) = 8$.

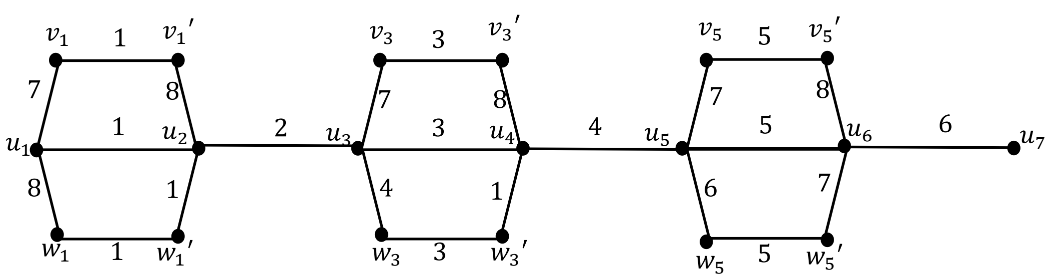

Figure 2.5. $AD(Q_7)$, rainbow 8-coloring

3. CONCLUSION

Based on the results of this research, we conclude that:

1. Rainbow connection number of double quadrilateral snake graph is given by:
$$rc(D(Q_n)) = n + 1.$$
2. Rainbow connection number of alternate double quadrilateral snake graph is given by:
$$rc(AD(Q_n)) = \begin{cases} 3, & if\ n = 3, \\ n+1, & if\ n\ is\ even, n \geq 2\ and\ if\ n\ is\ odd, n > 3. \end{cases}$$

REFERENCES

[1] Balakrishnan, R., & Ranganathan, K., 2012. A Textbook of Graph Theory Second Edition. New York: Springer.

[2] Chartrand, G., Lesniak, L., 1996. Graphs and Digraphs Third Edition. Chapman and Hall/CRC, London–New York–Washington, D.C.

[3] Chartrand, G., Johns, G. L., McKeon, K. A., & Zhang, P., 2008. Rainbow connection in graphs. *Mathematica bohemica*, *133*(1), 85-98.

[4] Gross, J.L, et al., 2019. Graphs and Theory Applications. Taylor & Francis Group, LLC.

[5] Harris, J.M, et al., 2000. Combinatorics and Graph Theory. New York: Springer.

[6] Maulani, A., Pradini, S., Setyorini, D., & Sugeng, K.A., 2019. Rainbow Connection Number of Cm ⊙ Pn and Cm ⊙ Cn. *Indonesian Journal of Combinatorics 3*,02, 95–108.

[7] Parmar, D, et al., 2019. Rainbow Connection Number of H- Graph. *Journal of Applied Science and Computations.*Vol 4, hlm 1487-1492.

[8] Parmar, D, et al., 2019. Rainbow Connection of Tringular Snake Graph. *Journal of Emerging Technologies and Innovative Research* (JETR), vol 6, Issue III.

[9] Surbakti, N.M., & Sugeng K.A., 2019 .The Rainbow Connection Number of a Watermill Graph. *Journal of Combinatorics IOP Conf* **1211**: 01200.

[10] Suthar, et al., 2021. Rainbow Connection Number Of Some Graphs. *Journal of Applied Science and Computation.* Vol 18, no 6, hlm 2836-2875.

[11] Syafrizal, S, et al., 2013. The Rainbow Connection of Fan and Sun. *Apllied Mathematical Sciences*, vol 7,no 64. hlm 3156-3159.

[12] Syafrizal, S, et al., 2014. Rainbow Connection Number of Some Graphs. *Apllied Mathematical Sciences*, vol 8,no 94. hlm 4693-4696.