JURNAL MATEMATIKA, STATISTIKA & KOMPUTASI E-ISSN: 2614-8811
https://journal.unhas.ac.id/index.php/jmsk/index
Vol. 18, No. 2, 181-197, January, 2022
DOI:10.20956/j.v18i2.17994

BEST PROXIMITY POINT THEOREMS FOR $\alpha^+F, (\theta-\phi)$-PROXIMAL CONTRACTION

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Abstract

In this paper, inspired by the idea of Suzuki type α^+F-proximal contraction in metric spaces, we prove a new existence of best proximity point for Suzuki type α^+F-proximal contraction and $\alpha^+(\theta-\phi)$-proximal contraction defined on a closed subset of a complete metric space. Our theorems extend, generalize, and improve many existing results.

Keywords: proximity point, α^+F-proximal contraction, $\alpha^+(\theta-\phi)$-proximal contraction.

1. Introduction and preliminaries

Best proximity point theorem analyses the condition under which the optimisation problem, namely, $\inf_{x\in A} d(x,Tx)$, has a solution. The point x is called the best proximity $(BPP(T)$ of $T: A \to B$, if $d(x,Tx) = d(A,B)$, where $\{d(A,B) = \inf d(x,y) : x \in A, y \in B\}$. Note that the best proximity point reduces to a fixed point if T is a self-mapping.

Sankar Raj [4] and Zhang et al. [5] defined the notion of P-property and weak P-property respectively. Hussain et al. [2] defined the concept of α^+-proximal admissible for non self mapping and introduced Suzuki type$\alpha^+\psi$- proximal contraction to generalize several best proximity results and obtained some best proximity point theorems for self-mappings.

Definition 1.1. [1]. Let (A,B) be a pair of non empty subsets of a metric space (X,d). We adopt the following notations:

$d(A,B) = \{\inf d(a,b) : a \in A, b \in B\}$;

$A_0 = \{\ a \in A$ there exists $b \in A$ such that $d(a,b) = d(A,B)\}$;

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$B_0 = \{\ b \in B$ there exists $a \in A$ such that $d(a,b) = d(A,B)\}$.

Definition 1.2. [1]. Let $T : A \to B$ be a mapping. An element x^* is said to be a best proximity point of T if

$$d(x^*, Tx^*) = d(A,B).$$

Definition 1.3. [2]. Let $\alpha : A \times A \to]-\infty, +\infty[$. We say that T is said to be α^+ proximal admissible if

$$\begin{cases} \alpha(x_1,x_2) \geq 0 \\ d(u_1, Tx_1) = d(A,B) \Rightarrow \alpha(u_1,u_2) \geq 0 \\ d(u_2, Tx_2) = d(A,B) \end{cases}$$

for all $x_1, x_2, u_1, u_2 \in A$.

Definition 1.4. [4]. Let (A,B) be a pair of non empty subsets of a metric space (X,d) such that A_0 is non empty. Then the pair (A,B) is to have P-property if and only

$$\begin{cases} d(x_1,y_1) = d(A,B) \\ d(x_2,y_2) = d(A,B) \end{cases} \Rightarrow d(x_1,x_2) = d(y_1,y_2)$$

for all $x_1, x_2 \in A_0$ *and* $y_1, y_2 \in B_0$.

Definition 1.5. [7]. Let $\mathcal{F}$ be the family of all functions $F\colon \mathbb{R}^+ \to \mathbb{R}$ such that

(F_1) F is strictly increasing;

(F_2) For each sequence $(x_n)_{n\in\mathbb{N}}$ of positive numbers

$$\lim_{n\to\infty} x_n = 0, \quad \text{if and only if} \quad \lim_{n\to\infty} F(x_n) = -\infty;$$

(F_3) There exists $k \in]0,1[$ such that $\lim_{x\to 0} x^k F(x) = 0$.

Definition 1.6. [3] Let Θ be the family of all functions $\theta :]0,+\infty[\to]1,+\infty[$ such that

(θ_1) θ is strictly increasing;

(θ_2) For each sequence $x_n \in]0,+\infty[$;

$$\lim_{n\to 0} x_n = 0, \quad \text{if and only if} \quad \lim_{n\to\infty} \theta(x_n) = 1;$$

(θ_3) θ is continuous.

Definition 1.7. [6] Let Φ be the family of all functions ϕ: $[1,+\infty[\ \to [1,+\infty[$, such that

(ϕ_1) ϕ is increasing;

(ϕ_2) For each $t \in]1,+\infty[$, $lim_{n\to\infty}\phi^n(t) = 1$;

(ϕ_3) ϕ is continuous.

Lemma 1.8. *If $\phi \in \Phi$ Then $\phi(1)$=1, and $\phi(t) < t$.*

Definition 1.9. [6]. Let (X, d) be a metric space and $T : X \to X$ be a mapping. T is said to be a $(\theta, \phi)-$contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for any $x, y \in X$,

$$d\left(Tx, Ty\right) > 0 \Rightarrow \theta\left[d\left(Tx, Ty\right)\right] \leq \phi\left[\theta\left(d\left(x, y\right)\right)\right],$$

2. Main Results

Now, we introduce the following concept which is a α^+F-proximal contraction and $\alpha^+(\theta, \phi)$-proximal contraction.

2.1. α^+F-proximal mapping.

Definition 2.1. The mapping $T : A \to B$ is called a Suzuki type α^+F-proximal contraction, if there exists $F \in \mathbb{F}$ and $\tau > 0$ such that

$$\frac{1}{2}d^*(x, Tx) \leq d(x, y) \Rightarrow \alpha(x, y) + F(d(Tx, Ty)) + \tau \leq F(M(x, y)) \tag{2.1}$$

for all $x, y \in A$, where $d^*(x, Tx) = d(x, Tx) - d(A, B)$, $\alpha : A \times A \to]-\infty, +\infty[$ and

$$M(x, y) = \left\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2} - d(A, B), \frac{d(x, Ty) + d(y, Tx)}{2} - d(A, B)\right\}$$

for all $x_1, x_2, u_1, u_2 \in A$.

Theorem 2.2. *Suppose A and B are nonempty closed subset of a complete metric space X with $A_0 \neq \varnothing$. Let $T : A \to B$ satisfy (2.1) together with the following assertions:*

(i) *$T(A_0) \in B_0$ and the pair (A, B) satisfies the weak P-property;*

(ii) *T is α^+-proximal admissible;*

(iii) *there exist elements $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq 0$;*

(iv) *T is continuous or*

(v) *F is continuous and A is α-regular, that $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x) \geq 0$ for all $n \in \mathbb{N}$.*

Then T has a best proximity point $z^* \in A$ such that $d(z^*, Tz^*) = d(A, B)$.

Proof. From condition (iii), there exist elements x_0, $x_1 \in A_0$ such that

$$d(x_1, Tx_0) = d(A, B) \quad and \quad \alpha(x_0, x_1) \geq 0.$$

Since $T(A_0) \in B_0$, there exists $x_2 \in A_0$ such that $d(x_2, Tx_1) = d(A, B)$.

Now, we have

$$d(x_2, Tx_1) = d(A, B), \alpha(x_1, x_2) \geq 0$$

Again, since $T(A_0) \in B_0$, there exists $x_3 \in A_0$ such that

$$d(x_3, Tx_2) = d(A, B).$$

Again since T is α^+-proximal admissible, this implies that $\alpha(x_2, x_3) \geq 0$. Thus, we have

$$d(x_3, Tx_2) = d(A, B) \quad and \quad \alpha(x_2, x_3) \geq 0.$$

Continuing this process, by induction, we construct a sequence $x_n \in A_0$ such that

$$d(x_{n+1}, Tx_n) = d(A, B) \quad and \quad \alpha(x_n, x_{n+1}) \geq 0, \forall n \in \mathbb{N}.$$

Since (A, B) satisfies the weak P property, we conclude from (2.1) that

$$d(x_n, x_{n+1}) \leq d(Tx_n, Tx_{n+1}), \forall n \in \mathbb{N}. \tag{2.2}$$

We shall prove that the sequence x_n is a Cauchy sequence. Let us first prove that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

By using the observations we can write

$$\begin{aligned}
\frac{1}{2}d^*(x_{n-1}, Tx_n) &= \frac{1}{2}d(x_{n-1}, Tx_n) - d(A, B) \\
&\leq \frac{1}{2}\left[d(x_{n-1}, x_n) + d(x_n, Tx_n)\right] - d(A, B) \\
&= \frac{1}{2}d(x_{n-1}, x_n) \\
&\leq d(x_{n-1}, x_n)
\end{aligned}$$

and

$$\begin{aligned}
M(x_{n-1},x_n) &= \max\left\{d(x_{n-1},x_n), \frac{d(x_{n-1},Tx_{n-1})+d(x_n,Tx_n)}{2}-d(A,B), \frac{d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})}{2}-d\right. \\
&\leq \max\left\{d(x_{n-1},x_n), \frac{d(x_{n-1},x_n)+d(x_n,Tx_{n-1})+d(x_n,x_{n+1})+d(x_{n+1},Tx_n)}{2}-d(A,B)\right\}, \\
&\left\{\frac{d(x_{n-1},x_{n+1})+d(x_{n+1},Tx_n)+d(x_n,Tx_{n-1})}{2}-d(A,B)\right\} \\
&\leq \max\left\{d(x_{n-1},x_n), \frac{d(x_{n-1},x_n)+d(x_n,Tx_{n-1})+d(x_n,x_{n+1})+d(x_{n+1},Tx_n)}{2}-d(A,B)\right\}, \\
&\left\{\frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})+d(x_{n+1},Tx_n)+d(x_n,Tx_{n-1})}{2}-d(A,B)\right\} \\
&= \max\left\{d(x_{n-1},x_n), \frac{d(x_{n-1},x_n)+d(A,B)+d(x_n,x_{n+1})+d(A,B)}{2}-d(A,B)\right\}, \\
&\left\{\frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})+d(A,B)+d(A,B)}{2}-d(A,B)\right\} \\
&= \max\left\{\frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{2}, \frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{2}\right\}, \\
&\left\{\frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})+d(A,B)+d(A,B)}{2}-d(A,B)\right\} \\
&\leq \max\left\{d(x_{n-1},x_n), d(x_n,x_{n+1})\right\}.
\end{aligned}$$

As T is α^+F-proximal contraction. Then

$$\begin{aligned}
F\left(d\left(x_n,x_{n+1}\right)\right) &\leq \tau + F\left(d\left(Tx_{n-1},Tx_n\right)\right) \\
&\leq \tau + F\left(d\left(Tx_{n-1},Tx_n\right)\right) + \alpha(x_{n-1},x_n) \\
&\leq F\left(M\left(x_{n-1},x_n\right)\right) \\
&\leq F\left(\max\left\{d(x_{n-1},x_n), d(x_n,x_{n+1})\right\}\right)
\end{aligned}$$

Now if $\max\left\{d(x_{n-1},x_n), d(x_n,x_{n+1})\right\} = d(x_n,x_{n+1})$, then

$$\begin{aligned}
F\left(d\left(Tx_{n-1},Tx_n\right)\right) &\leq F\left(d(x_n,x_{n+1}\right) + \tau \\
&< F\left(d(x_n,x_{n+1}\right).
\end{aligned}$$

which is a contradiction. Hence

$$\begin{aligned}
F\left(d\left(Tx_{n-1},Tx_n\right)\right) &\leq F\left(d(x_{n-1},x_n\right) - \tau \\
&\leq F\left(d(x_{n-2},x_{n-1}\right) - 2\tau \\
&\leq \ldots \leq F\left(d(x_0,x_1\right) - n\tau.
\end{aligned}$$

Taking the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} F(d\left(x_n, x_{n+1}\right)) = -\infty.$$

By (F_2), we obtain

$$\lim_{n\to\infty} d\left(x_n, x_{n+1}\right) = 0. \tag{2.3}$$

By condition (F_3) there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} d\left(x_n, x_{n+1}\right)^k d\left(x_n, x_{n+1}\right) = 0. \tag{2.4}$$

Since

$$F(d\left(x_n, x_{n+1}\right)) \leq F(d\left(x_0, x_1\right)) - n\tau,$$

we have

(2.5)

$$d\left(x_n, x_{n+1}\right)^k F(d\left(x_n, x_{n+1}\right)) \leq F(d\left(x_n, x_{n+1}\right))^k F(d\left(x_0, x_1\right)) - n\tau F(d\left(x_n, x_{n+1}\right))^k \leq 0.$$

Letting $n \longrightarrow +\infty$ in (2.5), we obtain

$$\lim_{n\to\infty} n\tau d\left(x_n, x_{n+1}\right)^k = 0.$$

From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$d\left(x_n, x_{n+1}\right) \leq \frac{1}{n^k}, \forall n \leq n_0.$$

Next we show that $\{x_n\}$ is a Cauchy sequence, i.e,

$$\lim_{n\to\infty} d\left(x_n, x_m\right) = 0 \ \forall m \in \mathbb{N}^*.$$

By triangular inequality, we have

$$\begin{aligned} d\left(x_n, x_m\right) &\leq d\left(x_n, x_{n+1}\right) + d\left(x_{n+1}, x_{+2}\right) + ... + d\left(x_{n+m-1}, x_{n+m}\right) \\ &\leq \frac{1}{n^k} + \frac{1}{(n+1)^k} + ... + \frac{1}{(n+m)^k} \\ &= \sum_{r=n}^{n+m-1} \frac{1}{(r)^k} \\ &\leq \sum_{r=1}^{\infty} \frac{1}{(r)^k}. \end{aligned}$$

Since $0 < k < 1$, $\sum\limits_{r=1}^{\infty} \frac{1}{(r)^k}$ is A convergent. Thus $d(x_n, x_m) \to 0$ as $n \to \infty$, which shows that $\{x_n\}$ is a Cauchy sequence. Then there exists $z \in X$ such that

$$\lim_{n\to\infty} d(x_n, z) = 0.$$

If (iv) holds, then

$$\lim_{n\to\infty} d(Tx_n, Tz) = 0.$$

and

$$d(A, B) = \lim_{n\to\infty} d(x_{n+1}, Tx_n) = d(z, Tz),$$

as required. Next, assume that (v) holds. Thus $\alpha(x_n, z) \geq 0$. If the flowing inequalities holds:

$$\frac{1}{2}d^*(x_n, Tx_n) > d(x_n, z) \ \ and \ \frac{1}{2}d^*(x_{n+1}, Tx_{n+1}) > d(x_{n+1}, z).$$

for some $n \in \mathbb{N}$, then by using (h) and definition of d^* , we obtain the following contraction:

$$\begin{aligned}
d(x_n, Tx_{n+1}) &\leq d(x_n, z) + d(z, Tx_{n+1}) \\
&< \frac{1}{2}\left[d^*(x_n, Tx_n) + d^*(x_{n+1}, Tx_{n+1})\right] \\
&= \frac{1}{2}\left[d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1}) - 2d(A, B)\right] \\
&\leq \frac{1}{2}\left[(x_n, x_{n+1}) + (x_{n+1}, Tx_n) + d(x_{n+1}, Tx_n) + d(Tx_n, Tx_{n+1}) - 2d(A, B)\right] \\
&= \frac{1}{2}\left[(x_n, x_{n+1}) + d(Tx_n, Tx_{n+1})\right] \\
&\leq (x_n, x_{n+1}).
\end{aligned}$$

Consequently, for any $n \in \mathbb{N}$, either

$$\frac{1}{2}d^*(x_n, Tx_n) \leq d(x_n, z) \ \ or \ \frac{1}{2}d^*(x_{n+1}, Tx_{n+1}) \leq d(x_{n+1}, z),$$

holds. Thus, we may pick a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\frac{1}{2}d^*(x_{n_k}, Tx_{n_k}) \leq d(x_{n_k}, z) \, and \, \alpha(x_{n_k}, z) \geq 0$$

for all $k \in \mathbb{N}$. By (2.1) we get

$$\begin{aligned}
F(d(Tx_{n_k}, Tz)) + \tau &\leq F(d(Tx_{n_k}, Tz)) + \tau + \alpha(x_{n_k}, z) \\
&\leq F\left[M(x_{n_k}, z)\right]
\end{aligned}$$

F is increasing, continuous function, we get

$$d\left(Tx_{n_k}, Tz\right) \leq M\left(x_{n_k}, z\right)$$

Notice that

$$\begin{aligned}
M(x_{n_k}z) &= \max\left\{d(x_{n_k}, z), \frac{d(x_{n_k}, Tx_{n_k}) + d(z, Tz)}{2} - d(A,B), \frac{d(x_{n_k}, Tz) + d(z, Tx_{n_k})}{2} - d(A,B)\right\} \\
&\leq \max\left\{d(x_{n_k}, z), \frac{d(x_{n_k}, x_{n_{k+1}}) + d(x_{n_{k+1}}, Tx_{n_k}) + d(z, Tz)}{2} - d(A,B)\right\}, \\
&\left\{\frac{d(x_{n_k}, z) + d(z, Tz) + d(z, x_{n_{k+1}}) + d(x_{n_{k+1}}, Tx_{n_k})}{2} - d(A,B)\right\} \\
&= \max\left\{d(x_{n_k}, z), \frac{d(x_{n_k}, x_{n_{k+1}}) + d(A,B) + d(z, Tz)}{2} - d(A,B)\right\}, \\
&\left\{\frac{d(x_{n_k}, z) + d(z, Tz) + d(z, x_{n_{k+1}}) + d(A,B)}{2} - d(A,B)\right\}.
\end{aligned}$$

which implies

$$\lim_{k\to\infty} M(x_{n_k}z) \leq \frac{d(z, x_{n_{k+1}}) + d(A,B)}{2}.$$

Further

$$\begin{aligned}
d(z, Tz) &\leq d(z, x_{n_{k+1}}) + d(x_{n_{k+1}}, Tx_{n_k}) + d(Tx_{n_k}, Tz) \\
&\leq d(z, x_{n_{k+1}}) + d(A,B) + d(Tx_{n_k}, Tz).
\end{aligned}$$

which gives

$$d(z, Tz) - d(z, x_{n_{k+1}}) - d(A,B) \leq d(Tx_{n_k}, Tz) \tag{2.6}$$

As $k \to \infty$ in (2.6) we deduce

$$d(z, Tz) - d(A,B) \leq \lim_{k\to\infty} d(Tx_{n_k}, Tz) \tag{2.7}$$

Therefore from (2.1), (2.6), and (2.7)

$$\begin{aligned}
d(z, Tz) - d(A,B) &\leq \lim_{k\to\infty} d(Tx_{n_k}, Tz) && (2.8) \\
&\leq \lim_{k\to\infty} M(x_{n_k}z) && (2.9) \\
&\leq \frac{d(z, x_{n_{k+1}}) + d(A,B)}{2}. && (2.10)
\end{aligned}$$

Now, if $d(z, Tz) – d(A,B) > 0$, then we get

$$d(z, Tz) - d(A,B) < \frac{d(z, Tz) - d(A,B)}{2},$$

a contradiction. Hence, $d(z, Tz) = d(A,B)$ as desired.

Example 2.3. Suppose $X = \mathbb{R}^2$ is equipped with the metric $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$, for all $(x_1, x_2), (y_1, y_2) \in X$. Let

$$A_1 = \{(x, y) \mid x = 1, 0 \leq y \leq \frac{1}{3}\};$$

$$A_2 = \{(x, y) \mid x = 3, y \geq 4\};$$

$$A_3 = \{(x, y) \mid x = 4, 0 \leq y \geq 3\}.$$

$A = A_1 \cup A_2 \cup A_3$ Further define

$$B_1 = \{(x, y) \mid x = \frac{1}{3}, \frac{1}{3} \leq y \leq 1\};$$

$$B_2 = \{(x, y) \mid x = 0, y \leq 3\};$$

$$B_3 = \{(x, y) \mid x = 3, y \geq 0\}$$

and $B = B_1 \cup B_2 \cup B_3$
Note that $d(A, B) = 1$, $A_0 = \{(x, y) \mid x = 1, 0 \leq y \leq \frac{1}{3}\}$ and $B_0 = \{(x, y) \mid x = \frac{1}{3}, \frac{1}{3} \leq y \leq 1\}$. Let, for $x_1 = (1, u_1)$, $x_2 = (1, u_2) \in A_0$ and $y_1 = (\frac{1}{3}, v_1)$, $y_2 = (1, v_2) \in B_0$, us have $d(x_1, y_1) = d(A, B) = 1$ and $d(x_2, y_2) = d(A, B) = 1$. Then

$$\frac{1}{3} + |u_1 - v_1| = 1$$

and

$$\frac{1}{3} + |u_2 - v_2| = 1$$

and so $|u_1 - v_1| = \frac{2}{3}$ and $|u_2 - v_2| = \frac{2}{3}$ Since $v_1, v_2 \geq u_1, u_2$, we have $v_1 = u_1 + \frac{2}{3}$ and $v_2 = u_2 + \frac{2}{3}$. This shows that $d(x_1, y_1) \leq d(x_2, y_2)$. So (A, B) satisfy the weak P-property.
Let $T : A \to B$ be defined by

$$T(x_1, x_2) = \begin{cases} (\frac{1}{3}, \frac{1}{3}) \ if \ x_1 = x_2 \\ (x_1, 0) \ if \ x_1 < x_2 \\ (0, x_2) \ if \ x_1 > x_2 \end{cases}$$

Notice that $T(A_0) \subseteq B_0$.
Define the functions $F :]0, +\infty[\to \mathbb{R}$ and $\alpha : A \times A \to \mathbb{R}$ by

$$F(t) = ln(t).$$

Then, $F \in \mathbb{F}$ and $\tau \in]0, +\infty[$ and

$$\alpha(x,y) = \begin{cases} 0 \; if \; x, y \in (1,0), (3,4), (4,3) \\ -\infty \; otherwise. \end{cases}$$

Let $\tau = \frac{1}{2}$. Assume that $\frac{1}{2}d^*(x, Tx) \leq d(x,y)$ and $\alpha(x,y) \geq 0 \; for x, y \in A$. then

$$\begin{cases} x = (1,0), \; x = (3,4) \; or \\ x = (1,0), \; x = (4,3) \; or \\ y = (1,0), \; x = (3,4) \; or \\ y = (1,0), \; x = (4,3) \end{cases}$$

Since $d(Tx, Ty) = d(Ty, Tx) \; and \; M(x,y) = M(y,x) \; for \; all \; x, y \in A$, we can suppose that

$$(x,y) = ((1,0),(3,4)) \; or \; (x,y) = ((1,0),(4,3)).$$

Now, we discuss the following cases:

$(i) \; if \; (x,y) = ((1,0),(3,4))$, then

$$\begin{aligned} F\left[d(Tx,Ty)\right] + \tau &= ln\left[d\left(T(1), T(0), (T(3), T(4)\right)\right] + \tau \\ &= ln(4) + \frac{1}{2} \\ &\leq ln(8) = ln\left[d\left(1, 0, (3,4)\right]\right. \\ &= F\left[d(x,y)\right] \\ &\leq F\left[M(x,y)\right]. \end{aligned}$$

$(ii) \; if \; (x,y) = ((1,0),(4,3))$, then

$$\begin{aligned} F\left[d(Tx,Ty)\right] + \tau &= ln\left[d\left(T(1), T(0), (T(4), T(3)\right)\right] + \tau \\ &= ln(4) + \frac{1}{2} \\ &\leq ln(8) = ln\left[d\left(1, 0, (4,3)\right]\right. \\ &= F\left[d(x,y)\right] \\ &\leq F\left[M(x,y)\right]. \end{aligned}$$

Consequently, we have $\frac{1}{2}d^*(x,Tx) \leq d(x,y) \Rightarrow F\left[d(Tx,Ty)\right] + \tau \leq F\left[M(x,y)\right]$. Thus all the assumptions of Theorem 2.2. are satisfied and $Bpp(T) = (1,0)$.

If $\alpha = 0$ on A, in Theorem 2.2, we obtain the following new result.

Corollary 2.4. *Suppose A and B are nonempty closed subset of a complete metric space X with $A_0 \neq \varnothing$. Let $T : A \to B$ satisfy the following assertions:*

(i) *$T(A_0) \in B_0$ and the pair (A, B) satisfies the weak P-property;*

(i) *$\frac{1}{2}d^*(x, Tx) \leq d(x, y) \Rightarrow F[d(Tx, Ty)] + \tau \leq F[M(x, y)]$*

Then T has a best proximity point $z^* \in A$ such that $d(z^*, Tz^*)=d(A, B)$.

2.2. $\alpha^+(\theta, \phi)$**-proximal contraction.**

Definition 2.5. The mapping $T : A \to B$ is called a Suzuki type $\alpha^+(\theta, \phi)$-proximal contraction, if there exists $\theta \in \Theta$ and $\phi \in \Phi$ such that

$$\frac{1}{2}d^*(x, Tx) \leq d(x, y) \Rightarrow \alpha(x, y) + \theta(d(Tx, Ty)) \leq \phi\left[\theta(M(x, y))\right] \tag{2.11}$$

for all $x, y \in A$, where $d^*(x, Tx) = d(x, Tx) - d(A, B)$, $\alpha : A \times A \to]-\infty, +\infty[$ and

$$M(x, y) = \left\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2} - d(A, B), \frac{d(x, Ty) + d(y, Tx)}{2} - d(A, B)\right\}$$

for all $x_1, x_2, u_1, u_2 \in A$.

Theorem 2.6. *Suppose A and B are nonempty closed subset of a complete metric space X with $A_0 \neq \varnothing$. Let $T : A \to B$ satisfy (2.11) together with the following assertions:*

(i) *$T(A_0) \in B_0$ and the pair (A, B) satisfies the weak P-property;*

(ii) *T is α^+-proximal admissible;*

(iii) *there exist elements $x_0, x_1 \in A$ such that $d(x_1, Tx_0) = d(A, B)$ and $\alpha(x_0, x_1) \geq 0$;*

(iv) *T is continuous or*

(v) *A is α-regular, that $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x) \geq 0$ for all $n \in \mathbb{N}$.*

Then T has a unique best proximity point $x^* \in A$ such that $d(x^*, Tx^*)=d(A, B)$.

Proof. As in the proof of Theorem 2.2, we can construct a sequence x_{n+1} satisfying

$$d(x_{n+1}, Tx_n) = d(A, B) \quad \textit{and} \quad \alpha(x_n, x_{n+1}) \geq 0, \forall n \in \mathbb{N}. \tag{2.12}$$

and

$$\frac{1}{2}d^*(x_{n-1}, Tx_{n-1}) \leq d(x_n, x_{n-1}) \quad \textit{and} \quad d(x_n, x_{n+1}) > 0, \forall n \in \mathbb{N}. \tag{2.13}$$

We shall prove that the sequence x_n is a Cauchy sequence. Let us first prove that

$$\lim_{n\to\infty} d\left(x_n, x_{n+1}\right) = 0.$$

By using the observations we can write

$$\begin{aligned}\frac{1}{2}d^*(x_{n-1}, Tx_n) &= \frac{1}{2}d(x_{n-1}, Tx_n) - d(A,B)\\ &\leq \frac{1}{2}\left[d(x_{n-1}, x_n) + d(x_n, Tx_n)\right] - d(A,B)\\ &= \frac{1}{2}d(x_{n-1}, x_n)\\ &\leq d(x_{n-1}, x_n)\end{aligned}$$

As in the proof of Theorem 2.2, we obtain

$$M(x_{n-1}, x_n) < \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\right\}.$$

As T is $\alpha^+(\theta,\phi)$-proximal contraction. Then

$$\begin{aligned}\theta\left(d\left(x_n, x_{n+1}\right)\right) &\leq \theta\left(d\left(Tx_{n-1}, Tx_n\right)\right)\\ &\leq \theta\left(d\left(Tx_{n-1}, Tx_n\right)\right) + \alpha(x_{n-1}, x_n)\\ &\leq \phi\left[\theta\left(M\left(x_{n-1}, x_n\right)\right)\right]\\ &\leq \phi\left[\theta\left(\max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\right\}\right)\right]\end{aligned}$$

Now if $\max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\right\} = d(x_n, x_{n+1})$, then

$$\begin{aligned}\theta\left(d\left(x_n, x_{n+1}\right)\right) &\leq \phi\left[\theta\left(d(x_n, x_{n+1})\right]\right.\\ &< \theta\left(d(x_n, x_{n+1}\right).\end{aligned}$$

which is a contradiction. Hence

$$\begin{aligned}\theta\left(d\left(x_n, x_{n+1}\right)\right) &\leq \phi\left[\theta\left(d(x_{n-1}, x_n)\right]\right.\\ &\leq \phi^2\left[\theta\left(d(x_{n-2}, x_{n-1})\right]\right.\\ &\leq \ldots \leq \phi^n\left[\theta\left(d(x_0, x_1)\right]\right..\end{aligned}$$

Taking the limit as $n \to \infty$, we have

$$1 \leq \theta(d\left(x_n, x_{n+1}\right)) \leq \lim_{n\to\infty} \phi^n\left[\theta(d\left(x_0, x_1\right))\right] = 1.$$

Since $\theta \in \Theta$, we obtain

$$\lim_{n\to\infty} d\left(x_n, x_{n+1}\right) = 0. \tag{2.14}$$

Next, we shall prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, i.e, $\lim_{n\to\infty} d\left(x_n, x_m\right) = 0$, for all $n \in \mathbb{N}$. Suppose to the contrary that exists $\varepsilon > 0$ and sequences $n_{(k)}$ and $m_{(k)}$ of natural numbers such that

$$m_{(k)} > n_{(k)} > k, \quad d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \geq \varepsilon, \quad D\left(x_{m_{(k)-1}}, x_{n_{(k)}}\right) < \varepsilon. \tag{2.15}$$

Using the triangular inequality, we find that,

$$\varepsilon \leq d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leq d\left(x_{m_{(k)}}, x_{n(k)-1}\right) + d\left(x_{n(k)-1}, x_{n_{(k)}}\right) \tag{2.16}$$

$$< \varepsilon + d\left(x_{n(k)-1}, x_{n_{(k)}}\right). \tag{2.17}$$

Then, by 2.15 and 2.16, it follows that

$$\lim_{k\to\infty} d\left(m_{(k)}, n_{(k)}\right) = \varepsilon. \tag{2.18}$$

Using the triangular inequality, we find that,

$$\varepsilon \leq d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leq d\left(x_{m_{(k)}}, x_{n(k)+1}\right) + d\left(x_{n(k)+1}, x_{n_{(k)}}\right) \tag{2.19}$$

and

$$\varepsilon \leq d\left(x_{m_{(k)}}, x_{n_{(k)+1}}\right) \leq d\left(x_{m_{(k)}}, x_{n(k)}\right) + d\left(x_{n(k)}, x_{n_{(k)+1}}\right) \tag{2.20}$$

Then, by (2.19) and (2.20), it follows that

$$\lim_{k\to\infty} d\left(m_{(k)}, n_{(k)+1}\right) = \varepsilon. \tag{2.21}$$

Similarly method, we conclude that

$$\lim_{k\to\infty} d\left(m_{(k)+1}, n_{(k)}\right) = \varepsilon. \tag{2.22}$$

Using again the triangular inequality,

$$d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) \leq d\left(x_{m_{(k)+1}}, x_{m_{(k)}}\right) + d\left(x_{m(k)}, x_{n_{(k)}}\right) + d\left(x_{n_{(k)}}, x_{n_{(k)+1}}\right). \tag{2.23}$$

On the other hand, using triangular inequality, we have

$$d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leq d\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right) + d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) + d\left(x_{n_{(k)+1}}, x_{n_{(k)}}\right). \tag{2.24}$$

Letting $k \to \infty$ in inequality (2.23) and (2.24), we obtain

$$\lim_{k\to\infty} d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) = \varepsilon. \tag{2.25}$$

Substituting $x = x_{m_{(k)}}$ and $y = x_{n_{(k)}}$ in assumption of the theorem, we get,

$$\theta\left(d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)\right) \leq \theta\left(d\left(Tx_{m_{(k)}}, Tx_{n_{(k)}}\right)\right) \leq \phi\left[\theta\left(M\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right)\right]. \tag{2.26}$$

and

$$\begin{aligned} M\left(x_{m_{(k)}}, x_{n_{(k)}}\right) &= \max\left\{d(x_{m_{(k)}}, x_{n_{(k)}}), \frac{d(x_{m_{(k)}}, Tx_{m_{(k)}}) + d(x_{n_{(k)}}, Tx_{n_{(k)}})}{2} - d(A,B)\right\}, \\ &\left\{\frac{d(x_{m_{(k)}}, Tx_{n_{(k)}}) + d(x_{n_{(k)}}, Tx_{m_{(k)}})}{2} - d(A,B)\right\} \\ &\leq \max\left\{d(x_{m_{(k)}}, x_{n_{(k)}})\right\}, \\ &\left\{\frac{d(x_{m_{(k)}}, x_{m_{(k)}+1}) + d(x_{m_{(k)}+1}, Tx_{m_{(k)}}) + d(x_{n_{(k)}}, x_{n_{(k)+1}}) + d(x_{n_{(k)}+1}, Tx_n)}{2} - d(A,B)\right\}, \\ &\left\{\frac{d(x_{m_{(k)}}, x_{n_{(k)}+1}) + d(x_{n_{(k)}+1}, Tx_{n_{(k)}}) + d(x_{n_{(k)}}, x_{m_{(k)+1}}) + d(x_{m_{(k)}+1}, Tx_m)}{2} - d(A,B)\right\} \\ &= \max\left\{d(x_{m_{(k)}}, x_{n_{(k)}}), \frac{d(x_{m_{(k)}}, x_{m_{(k)}+1}) + d(x_{n_{(k)}}, x_{n_{(k)+1}})}{2}, \frac{d(x_{m_{(k)}}, x_{n_{(k)}+1}) + d(x_{n_{(k)}}, x_{m_{(k}}}{2}\right. \end{aligned}$$

Passing the limit as $n \to +\infty$, we get

$$\lim_{k\to\infty} M\left(x_{m_{(k)}}, x_{n_{(k)}}\right) = \varepsilon$$

Letting Letting $k \to \infty$ in (2.26), and using (θ_1), (θ_3) , (ϕ_3) and Lemma (1.8) we obtain

$$\theta\left(\lim_{k\to\infty} d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)\right) \leq \phi\left[\theta \lim_{k\to\infty}\left(M\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right)\right].$$

We derive

$$\varepsilon < \varepsilon.$$

Which is a contradiction. Thus $\lim_{n,m\to\infty} d(x_n, x_m) = 0$, which shows that $\{x_n\}$ is a Cauchy sequence. Then there exists $z \in X$ such that

$$\lim_{n\to\infty} d(x_n, z) = 0.$$

If (iv) holds, then

$$\lim_{n\to\infty} d(Tx_n, Tz) = 0.$$

and

$$d(A,B) = \lim_{n\to\infty} d(x_{n+1}, Tx_n) = d(z, Tz),$$

Then T has a best proximity point.

Uniqueness. Now, suppose that $z^*, u^* \in A$ are two distinct best proximity points for T such that $z^* = u^*$. Since $d(x^*, Tz^*) = d(u^*, Tu^*) = d(A,B)$, using the P property, we conclude that

$$d(z^*, u^*) = d(Tz^*, Tu^*).$$

Since T is an $\alpha-$ proximal $(\theta - \phi)-$mapping, we obtain

$$\theta\left(d(Tz^*, Tu^*)\right) \leq \phi\left[\theta\left(d(z^*, u^*)\right)\right].$$

Therefore

$$\theta\left(d(A, B)\right) \leq \phi\left[\theta\left(d(A, B)\right)\right].$$

Then $d(A, B) < d(A, B)$, which is a contradiction. as required. Next, assume that (v) holds. Thus $\alpha(x_n, z) \geq 0$. As in the proof of Theorem (2.2), we can deduce there is a subsequence a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\frac{1}{2}d^*\left(x_{n_k}, Tx_{n_k}\right) \leq d\left(x_{n_k}, z\right) and\ \alpha(x_{n_k}, z) \geq 0$$

for all $k \in \mathbb{N}$. By (2.11) we get

$$\begin{aligned}\theta(d\left(Tx_{n_k}, Tz\right)) &\leq \theta(d\left(Tx_{n_k}, Tz\right)) + \alpha(x_{n_k}, z)\\ &\leq \phi\left[\theta\left(M\left(x_{n_k}, z\right)\right)\right]\\ &< \theta\left(M\left(x_{n_k}, z\right)\right).\end{aligned}$$

θ is increasing, we get

$$d\left(Tx_{n_k}, Tz\right) \leq M\left(x_{n_k}, z\right),$$

which implies

$$\lim_{k\to\infty} M(x_{n_k} z) \leq \frac{d(z, x_{n_{k+1}}) + d(A, B)}{2}. \tag{2.27}$$

Further

$$\begin{aligned}d(z, Tz) &\leq d(z, x_{n_{k+1}}) + d(x_{n_{k+1}}, Tx_{n_k}) + d(Tx_{n_k}, Tz)\\ &\leq d(z, x_{n_{k+1}}) + d(A, B) + d(Tx_{n_k}, Tz).\end{aligned}$$

which gives

$$d(z, Tz) - d(z, x_{n_{k+1}}) - d(A, B) \leq d(Tx_{n_k}, Tz) \tag{2.28}$$

As $k \to \infty$ in (2.28) we deduce

$$d(z, Tz) - d(A, B) \leq \lim_{k\to\infty} d(Tx_{n_k}, Tz) \tag{2.29}$$

Therefore from (2.27), (2.28), and (2.29)

$$d(z,Tz)-d(A,B) \le \lim_{k\to\infty} d(Tx_{n_k},Tz) \tag{2.30}$$

$$\le \lim_{k\to\infty} M(x_{n_k}z) \tag{2.31}$$

$$\le \frac{d(z,x_{n_{k+1}})+d(A,B)}{2}. \tag{2.32}$$

Now, if $d(z,Tz)$–$d(A,B)>0$, then we get

$$d(z,Tz)-d(A,B) < \frac{d(z,Tz)-d(A,B)}{2},$$

a contradiction. Hence, $d(z,Tz)=d(A,B)$ as desired. □

□

Definition 2.7. [2] The mapping $T:A\to B$ is called a Suzuki type $\alpha^+(\theta)$-proximal contraction, if where $\alpha : A\times A\to\,]-\infty,+\infty[$, if there exists $\theta\in\Theta$ and $k\in\,]0,1[$ such that

$$\frac{1}{2}d^*(x,Tx)\le d(x,y)\Rightarrow \alpha(x,y)+\theta(d(Tx,Ty))\le[\theta(M(x,y))]^k \tag{2.33}$$

for all $x,y\in A$, where $d^*(x,Tx)=d(x,Tx)-d(A,B)$, $\alpha : A\times A\to\,]-\infty,+\infty[$ and

$$M(x,y)=\left\{d(x,y),\frac{d(x,Tx)+d(y,Ty)}{2}-d(A,B),\frac{d(x,Ty)+d(y,Tx)}{2}-d(A,B)\right\}$$

for all $x_1,x_2,u_1,u_2\in A$.

If $\phi(t)=t^k$, in Theorem 2.6, we obtain the following new result.

Corollary 2.8. [2] *Suppose A and B are nonempty closed subset of a complete metric space X with $A_0\neq\varnothing$. Let $T:A\to B$ satisfy (2.11) together with the following assertions:*

(i) *$T(A_0)\in B_0$ and the pair (A,B) satisfies the weak P-property;*
(ii) *T is α^+-proximal admissible;*
(iii) *there exist elements $x_0,x_1\in A$ such that $d(x_1,Tx_0)=d(A,B)$ and $\alpha(x_0,x_1)\ge 0$;*
(iv) *T is continuous or*
(v) *A is α-regular, that $\{x_n\}$ is a sequence in A such that $\alpha(x_n,x)\ge 0$ for all $n\in\mathbb{N}$.*

Then T has a unique best proximity point $x^*\in A$ such that $d(x^*,Tx^*)=d(A,B)$.

If $\alpha = 0$ on A, in Theorem 2.6, we obtain the following new result.

Corollary 2.9. *Suppose A and B are nonempty closed subset of a complete metric space X with $A_0 \neq \varnothing$. Let $T : A \to B$ satisfy the following assertions:*

(i) $T(A_0) \in B_0$ *and the pair (A, B) satisfies the weak P-property;*

(ii) $\frac{1}{2}d^*(x, Tx) \leq d(x, y) \Rightarrow \theta[d(Tx, Ty)] \leq \phi[\theta(M(x, y))]$

Then T has a best proximity point $z^* \in A$ such that $d(z^*, Tz^*) = d(A, B)$.

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